

Gauge and matter superfield theories on S^2

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ABSTRACT: We develop a superfield formulation of gauge and matter field theories on a two-dimensional sphere with rigid $\mathcal{N} = (2, 2)$ as well as extended supersymmetry. The construction is based on a supercoset $\frac{SU(2|1)}{U(1) \times U(1)}$ containing S^2 as the bosonic subspace. We derive an explicit form of supervielbein and covariant derivatives on this coset, and use them to construct classical superfield actions for gauge and matter supermultiplets in this superbackground. We then apply superfield methods for computing one-loop partition functions of these theories and demonstrate how the localization technique works directly in the superspace.

KEYWORDS: Supersymmetric gauge theory, Extended Supersymmetry, Superspaces, Supergravity Models

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1 Introduction and summary

The method of supersymmetric localization has proved to be a very powerful tool for computing various quantum quantities such as partition functions, Wilson loops or correlation functions exactly, at all orders in perturbation theory (see, e.g., [1, 2] for reviews). Originally used for four-dimensional supersymmetric gauge theories in [3], recently this method has also been applied to study various non-perturbative aspects of two-dimensional supersymmetric field theories. In particular, quantum partition functions of two-dimensional $\mathcal{N} = (2, 2)$ supergauge models on S^2 were computed in [4, 5] and used for studying Seiberg-like dualities of these models [6]. Some mathematical aspects of such dualities were investigated in a recent paper [7]. In [8–10] it was shown that partition functions of $\mathcal{N} = (2, 2)$ gauge theories on S^2 compute exact Kähler potentials for Calabi-Yau target spaces of $\mathcal{N} = (2, 2)$ non-linear sigma-models. A systematic construction of supersymmetric backgrounds as solutions of the $\mathcal{N} = (2, 2)$ supergravity was given recently in [11]. Some of the above mentioned results were extended to the two-dimensional manifolds with boundaries in [12–14].

To apply the supersymmetric localization techniques one puts classical actions for supersymmetric field theories on a compact manifold with rigid supersymmetry, such as a sphere. A systematic prescription for constructing such actions was given in [15]: one should couple the gauge and matter field models to off-shell supergravity theories and then fix the supergravity background to be e.g. a supersymmetric sphere or an AdS space. In the limit of large Planck mass, the supergravity fields decouple and one is left with a Lagrangian for the field theory on a curved background with rigid supersymmetry. This procedure is equivalent to considering a superfield supergravity coupled to matter superfields which include all necessary auxiliary fields. Once the supergravity background is fixed, one automatically gets superfield theories which respect all (super)symmetries of the background, see, e.g., [16, 17]. However, off-shell supersymmetry formulations of supergravity are not always available. Therefore, in some cases, alternative methods should be used for the construction of actions for supersymmetric fields on curved backgrounds which do not require the knowledge of supergravity.

In a recent paper [18] we applied superfield techniques for constructing actions for various supersymmetric models on S^3 and computing their partition functions. These superfield models were formulated on the supercoset $\frac{\text{SU}(2|1)}{\text{U}(1)}$ containing S^3 as its bosonic body. The aim of this paper is to introduce, in a similar way, a suitable curved superspace for supersymmetric gauge and matter field theories on the two-sphere and develop an approach for studying their quantum properties directly in the superspace.

The two-dimensional $(2, 2)$ superfield supergravity was studied in a series of papers [19–21] and corresponding matter superfield theories were coupled to $d = 2$ supergravity in [22]. Basically, in this paper we take a particular solution of the $\mathcal{N} = (2, 2)$ $d = 2$ supergravity corresponding to the Wick rotated counterpart of AdS_2 space, i.e. the two-sphere S^2 , and consider various matter superfield theories on such a superbackground. The supergravity solution of our interest is the supercoset $\frac{\text{SU}(2|1)}{\text{U}(1) \times \text{U}(1)}$ which contains the two-sphere as its bosonic body. Note that $\text{SU}(2|1)$ is the minimal possible supersymmetry group for the

theories on S^2 since the two-component spinors on S^2 are complex. In the next section, we construct the Cartan forms on the supercoset $\frac{\text{SU}(2|1)}{\text{U}(1) \times \text{U}(1)}$ and use them to define supercovariant derivatives, supertorsion and supercurvature. These objects describe the geometry of the background $\mathcal{N} = (2, 2)$ superspace in which gauge and matter superfields propagate and are used to construct classical superfield actions on $\frac{\text{SU}(2|1)}{\text{U}(1) \times \text{U}(1)}$ given in section 3.

The use of superfields on $\frac{\text{SU}(2|1)}{\text{U}(1) \times \text{U}(1)}$ also allows us to construct Lagrangians for models with extended supersymmetry. In section 4 we present classical actions for an $\mathcal{N} = (4, 4)$ hypermultiplet, and $\mathcal{N} = (4, 4)$ and $\mathcal{N} = (8, 8)$ SYM fields on S^2 . We also consider theories obtained by the reduction to S^2 of the $d = 3$ Gaiotto-Witten [23] and ABJM [24] models. For all these models we derive superfield transformations under extended (hidden) supersymmetry which does not belong to $\text{SU}(2|1)$. All models with extended supersymmetry involve chiral superfields which can have, in principle, different charges associated to the $\text{U}(1)$ R-symmetry generator of the group $\text{SU}(2|1)$. We find constraints on the values of the R-charges of the chiral superfields imposed by the extended supersymmetry.

As we will demonstrate, the superfield formulation is useful not only for constructing classical actions for supersymmetric field theories on S^2 , but also for computing their partition functions. In section 5 we show how the one-loop partition functions for gauge and matter superfields computed in [4, 5] can be derived with the use of the superspace methods which make the cancellations between bosonic and fermionic contributions automatic. Another advantage of the superfield approach is the trivialization of the procedure of finding critical points around which the functional integrals localize. On the Coulomb branch they simply correspond to constant vacuum values of gauge superfield strengths. In section 6 we demonstrate how the standard localization formulas for the partition functions of $\mathcal{N} = (2, 2)$ gauge theories [4, 5] appear from functional integrals over gauge superfields on $\frac{\text{SU}(2|1)}{\text{U}(1) \times \text{U}(1)}$. We apply the localization method for deriving partition functions of the Gaiotto-Witten and ABJM models reduced to S^2 .

An important feature of the two-dimensional $(2, 2)$ supersymmetric theories is the possibility of having not only conventional chiral and gauge superfields, but also their twisted counterparts [25–27]. Quantum partition functions of models on S^2 with twisted supermultiplets were studied in [8, 9]. In the present paper we restrict ourselves by considering only ordinary $\mathcal{N} = (2, 2)$ multiplets which have four-dimensional analogs. Superspace study of partition functions of models with twisted supermultiplets on S^2 will be given elsewhere.

We keep the structure of this paper close to the previous one [18] and use most of the superspace conventions introduced therein.

2 $\frac{\text{SU}(2|1)}{\text{U}(1) \times \text{U}(1)}$ supergeometry

2.1 $\text{SU}(2|1)$ superalgebra

The two-dimensional sphere S^2 appears as the bosonic body of the supercoset $\frac{\text{SU}(2|1)}{\text{U}(1) \times \text{U}(1)}$. The supersymmetry $\text{SU}(2|1)$ of this supermanifold is generated by the Grassmann-even $\text{SU}(2)$ generators $J_{\underline{a}} = (J_1, J_2, J_3)$ and the $\text{U}(1)$ generator R , and by the Grassmann-odd supercharges Q_{α} and $\bar{Q}_{\dot{\alpha}}$ ($\alpha = 1, 2$). They obey the following non-zero (anti)commutation

relations

$$\begin{aligned} [J_{\underline{a}}, J_{\underline{b}}] &= i\varepsilon_{\underline{a}\underline{b}\underline{c}}J_{\underline{c}}, & [J_{\underline{a}}, Q_{\alpha}] &= -\frac{1}{2}(\gamma_{\underline{a}})_{\alpha}^{\beta}Q_{\beta}, & [J_{\underline{a}}, \bar{Q}_{\alpha}] &= -\frac{1}{2}(\gamma_{\underline{a}})_{\alpha}^{\beta}\bar{Q}_{\beta}, \\ \{Q_{\alpha}, \bar{Q}_{\beta}\} &= \gamma_{\alpha\beta}^{\underline{a}}J_{\underline{a}} + \frac{1}{2}\varepsilon_{\alpha\beta}R, & [R, Q_{\alpha}] &= -Q_{\alpha}, & [R, \bar{Q}_{\alpha}] &= \bar{Q}_{\alpha}. \end{aligned} \quad (2.1)$$

Here $(\gamma_{\underline{a}})_{\alpha}^{\beta}$ are three-dimensional gamma-matrices which can be taken to be equal to the Pauli matrices.¹

In the $su(2|1)$ superalgebra, Q_{α} and \bar{Q}_{α} are related by the complex conjugation $(Q_{\alpha})^* = \varepsilon^{\alpha\beta}\bar{Q}_{\beta}$. However, the Wick rotated Lagrangians on S^2 are not supposed to be real, so, in general, we will consider S^2 (super)fields like Φ and $\bar{\Phi}$ as independent ones. We denote the number of components of the supersymmetry generators Q_{α} and \bar{Q}_{α} by $\mathcal{N} = (2, 2)$. We employ this notation to indicate the number of supersymmetries on S^2 by analogy with supersymmetries in 2d spaces of Lorentz signature.

It is convenient to split the $SU(2)$ generators $J_{\underline{a}}$ into the S^2 -boosts $J_a = (J_1, J_2)$ and the $U(1)$ -generator J_3 and then perform the re-scaling of the $SU(2|1)$ generators with the S^2 radius r ,

$$J_a \rightarrow P_a = \frac{J_a}{r}, \quad J_3 \rightarrow M = J_3, \quad Q_{\alpha} \rightarrow \sqrt{r}Q_{\alpha}, \quad \bar{Q}_{\alpha} \rightarrow \sqrt{r}\bar{Q}_{\alpha}. \quad (2.2)$$

In terms of these generators the (anti)commutation relations of the $su(2|1)$ superalgebra (2.1) take the form

$$\begin{aligned} [P_a, P_b] &= \frac{i}{r^2}\varepsilon_{ab}M, & [M, P_a] &= i\varepsilon_{ab}P_b, \\ [P_a, Q_{\alpha}] &= -\frac{1}{2r}(\gamma_a)_{\alpha}^{\beta}Q_{\beta}, & [P_a, \bar{Q}_{\alpha}] &= -\frac{1}{2r}(\gamma_a)_{\alpha}^{\beta}\bar{Q}_{\beta}, \\ [M, Q_{\alpha}] &= -\frac{1}{2}(\gamma_3)_{\alpha}^{\beta}Q_{\beta}, & [M, \bar{Q}_{\alpha}] &= -\frac{1}{2}(\gamma_3)_{\alpha}^{\beta}\bar{Q}_{\beta}, \\ \{Q_{\alpha}, \bar{Q}_{\beta}\} &= \gamma_{\alpha\beta}^aP_a + \frac{1}{r}\gamma_{\alpha\beta}^3M + \frac{1}{2r}\varepsilon_{\alpha\beta}R, & [R, Q_{\alpha}] &= -Q_{\alpha}, \quad [R, \bar{Q}_{\alpha}] = \bar{Q}_{\alpha}. \end{aligned} \quad (2.3)$$

The meaning of this re-scaling is that in the limit $r \rightarrow \infty$ the (anti)commutation relations (2.3) reduce to the $d = 2$ Euclidean flat space superalgebra in which P_a play the role of the momenta operators while M stands for the angular momentum.

We will use the $SU(2|1)$ (anti)commutation relations (2.3) for constructing the Cartan forms on the supercoset $\frac{SU(2|1)}{U(1) \times U(1)}$ in a simple chiral basis realization which is the most suitable for our purposes. A different realization of the Cartan forms on this coset space was given in [28, 29] in the study of a supersymmetric Landau model.

2.2 Supervielbein and $U(1)$ -connections

Let $z^M = (x^m, \theta^{\mu}, \bar{\theta}^{\mu})$, $m = 1, 2$, $\mu = 1, 2$, be local coordinates on the supercoset $\frac{SU(2|1)}{U(1) \times U(1)}$. In a given coordinate system the supervielbein on $\frac{SU(2|1)}{U(1) \times U(1)}$ is described by a set of one-forms

$$E^A = dz^M E_M^A(z), \quad E^A = (E^a, E^{\alpha}, \bar{E}^{\alpha}). \quad (2.4)$$

¹We use the following conventions for the antisymmetric tensors with vector ε_{ab} and spinor $\varepsilon_{\alpha\beta}$ indices: $\epsilon_{12} = \epsilon^{12} = 1$, $\varepsilon_{12} = -\varepsilon^{12} = 1$. The spinor indices are raised and lowered by the rules $\theta_{\alpha} = \varepsilon_{\alpha\beta}\theta^{\beta}$, $\theta^{\alpha} = \varepsilon^{\alpha\beta}\theta_{\beta}$ and are contracted as follows $\theta^2 = \theta^{\alpha}\theta_{\alpha}$, $\bar{\theta}^2 = \bar{\theta}^{\alpha}\bar{\theta}_{\alpha}$.

They are components of the Maurer-Cartan form $J = G^{-1}dG$

$$J = G^{-1}dG = iE^a P_a + iE^\alpha Q_\alpha + i\bar{E}^\alpha \bar{Q}_\alpha + i\Omega_{(M)}M + i\Omega_{(R)}R, \quad (2.5)$$

where $G(x, \theta, \bar{\theta})$ is a representative of the supercoset $\frac{\text{SU}(2|1)}{\text{U}(1) \times \text{U}(1)}$, and $\Omega_{(M)}$ and $\Omega_{(R)}$ are U(1)-connection one-forms corresponding to the M and R generators of $\text{SU}(2|1)$, respectively.

To find an explicit form of the supervielbein we consider the following parametrization of the coset representative

$$G = b(x)f(\theta, \bar{\theta}), \quad b(x) = e^{ix^m P_m}, \quad f(\theta, \bar{\theta}) = e^{i\theta^\alpha Q_\alpha} e^{i\bar{\theta}^\beta \bar{Q}_\beta}. \quad (2.6)$$

Then

$$G^{-1}dG = f^{-1}(d + ie^a(x)P_a + i\omega(x)M)f, \quad (2.7)$$

where $e^a(x) = dx^m e_m^a(x)$ and $\omega(x) = dx^m \omega_m(x)$ are bosonic zweibein and the U(1) connection on $S^2 = \text{SU}(2)/\text{U}(1)$. They obey the torsion-less constraint and determine the round-sphere curvature

$$de^a + \omega^{ab} \wedge e^b = 0, \quad d\omega^{ab} = \frac{1}{r^2} e^a \wedge e^b, \quad (2.8)$$

where $\omega^{ab} = \epsilon^{ab} \omega$. Note that the indices a, b, \dots are raised and lowered with the delta-symbol δ_{ab} due to the Euclidian signature.

Now, applying the algebra (2.3) we find the explicit expressions for the components of the supervielbein and superconnections in the decomposition (2.5),

$$\begin{aligned} E^\alpha &= d\theta^\alpha, \\ \bar{E}^\alpha &= d\bar{\theta}^\alpha - \frac{1}{2r} d\theta^\alpha \bar{\theta}^2, \\ E^a &= e^a(x) - i d\theta^\alpha \bar{\theta}^\beta \gamma_{\alpha\beta}^a, \\ \Omega_{(M)} &= \omega(x) - \frac{i}{r} d\theta^\alpha \gamma_{\alpha\beta}^3 \bar{\theta}^\beta, \\ \Omega_{(R)} &= -\frac{i}{2r} d\theta^\alpha \bar{\theta}_\alpha, \end{aligned} \quad (2.9)$$

where \mathbf{d} is the Killing-spinor covariant differential

$$\mathbf{d}\theta^\alpha = d\theta^\alpha - \frac{i}{2r} (\gamma_a)_\beta^\alpha \theta^\beta e^a - \frac{i}{2} (\gamma_3)_\beta^\alpha \theta^\beta \omega, \quad \mathbf{d}^2 = 0. \quad (2.10)$$

Note that the $\frac{\text{SU}(2|1)}{\text{U}(1) \times \text{U}(1)}$ supergeometry constructed in this way has a smooth flat limit at $r \rightarrow \infty$.

The inverse supervielbein is given by a set of differential operators

$$E_A = E_A^M \partial_M, \quad \partial_M = (\partial_m, \partial_\alpha, \bar{\partial}_\alpha) \quad (2.11)$$

such that

$$E_A^M E_M^B = \delta_A^B. \quad (2.12)$$

For instance, in the coordinate system in which the supervielbein E^A is given by (2.9) we have the following explicit expressions for the components of E_A :

$$\begin{aligned} E_a &= \partial_a + \left(\frac{i}{2r} (\gamma_a)_\beta^\alpha + \frac{i}{2} \omega_a (\gamma_3)_\beta^\alpha \right) \left(\theta^\beta \partial_\alpha + \bar{\theta}^\beta \bar{\partial}_\alpha \right), \\ E_\alpha &= \partial_\alpha + i \gamma_{\alpha\beta}^a \bar{\theta}^\beta \partial_a + \left[\frac{1}{2r} (\gamma^a)_\gamma^\beta (\gamma^a)_{\alpha\delta} + \frac{1}{2} (\gamma_3)_\gamma^\beta (\gamma^a)_{\alpha\delta} \omega_a \right] \theta^\gamma \bar{\theta}^\delta \partial_\beta + \frac{i}{4} \omega_a \epsilon^{ab} \left(\gamma^b \right)_\alpha^\beta \bar{\theta}^2 \bar{\partial}_\beta, \\ \bar{E}_\alpha &= \bar{\partial}_\alpha, \end{aligned} \quad (2.13)$$

where $\partial_a = e_a^m(x) \partial_m$ and $\omega_a = e_a^m \omega_m(x)$ are purely bosonic.

The explicit form of the supervielbein (2.9) allows us to find its Berezinian,

$$E \equiv \text{Ber} E_M^A = \det e_m^a(x) = \sqrt{h(x)}, \quad (2.14)$$

where $h(x) = \det h_{mn}(x)$ and $h_{mn}(x)$ is a metric on S^2 . The Berezinian (2.14) appears to be independent of the Grassmann variables in the coordinate system corresponding to the choice of the coset representative (2.6). As a consequence, the supervolume of the coset $\frac{\text{SU}(2|1)}{\text{U}(1) \times \text{U}(1)}$ vanishes

$$\int d^2 x d^2 \theta d^2 \bar{\theta} E = 0. \quad (2.15)$$

We stress that this is the coordinate independent property of this supermanifold.

2.3 Covariant differential, torsion and curvature

By construction, the differential form (2.5) obeys the Maurer-Cartan equation

$$dJ + J \wedge J = 0, \quad (2.16)$$

which implies a number of relations for the components of the supervielbein and superconnections:

$$\begin{aligned} dE^a + \epsilon^{ab} \Omega_{(M)} \wedge E^b - i E^\alpha \wedge \bar{E}^\beta \gamma_{\alpha\beta}^a &= 0, \\ d\Omega_{(M)} - \frac{1}{2r^2} E^a \wedge E^b \epsilon_{ab} - \frac{i}{r} E^\alpha \wedge \bar{E}^\beta \gamma_{\alpha\beta}^3 &= 0, \\ d\Omega_{(R)} - \frac{i}{2r} \varepsilon_{\alpha\beta} E^\alpha \wedge \bar{E}^\beta &= 0, \\ dE^\alpha - i \Omega_{(R)} \wedge E^\alpha - \frac{i}{2r} E^a \wedge E^\beta (\gamma_a)_\beta^\alpha - \frac{i}{2} \Omega_{(M)} \wedge E^\beta (\gamma_3)_\beta^\alpha &= 0, \\ d\bar{E}^\alpha + i \Omega_{(R)} \wedge \bar{E}^\alpha - \frac{i}{2r} E^a \wedge \bar{E}^\beta (\gamma_a)_\beta^\alpha - \frac{i}{2} \Omega_{(M)} \wedge \bar{E}^\beta (\gamma_3)_\beta^\alpha &= 0. \end{aligned} \quad (2.17)$$

These equations can be recast in the unified form

$$\mathcal{D}E^A = dE^A + \Omega^{AB} \wedge E^B = T^A, \quad (2.18)$$

where T^A is the supertorsion with components

$$\begin{aligned} T^a &= i \gamma_{\alpha\beta}^a E^\alpha \wedge \bar{E}^\beta, \\ T^\alpha &= \frac{i}{2r} (\gamma_a)_\beta^\alpha E^a \wedge E^\beta, \\ \bar{T}^\alpha &= \frac{i}{2r} (\gamma_a)_\beta^\alpha E^a \wedge \bar{E}^\beta \end{aligned} \quad (2.19)$$

and $\mathcal{D} = d + \Omega$ is the covariant differential constructed with the superconnection Ω^{AB} . Non-vanishing components of the latter are

$$\begin{aligned}\Omega^{ab} &= \epsilon^{ab} \Omega_{(M)}, \\ \Omega_{\beta}^{\alpha} &= -i\delta_{\beta}^{\alpha} \Omega_{(R)} - \frac{i}{2}(\gamma_3)_{\beta}^{\alpha} \Omega_{(M)}, \\ \bar{\Omega}_{\beta}^{\alpha} &= i\delta_{\beta}^{\alpha} \Omega_{(R)} - \frac{i}{2}(\gamma_3)_{\beta}^{\alpha} \Omega_{(M)}.\end{aligned}\tag{2.20}$$

Note that the superconnection Ω^{AB} is Abelian. Hence, the corresponding super-curvature is simply

$$\mathcal{R}^{AB} = d\Omega^{AB},\tag{2.21}$$

or explicitly,

$$\begin{aligned}\mathcal{R}^{ab} &= d\Omega^{ab} = \frac{1}{r^2} E^a \wedge E^b + \frac{i}{r} \epsilon^{ab} \gamma_{\alpha\beta}^3 E^{\alpha} \wedge \bar{E}^{\beta}, \\ \mathcal{R}_{\beta}^{\alpha} &= d\Omega_{\beta}^{\alpha} = \frac{1}{2r} \left[\delta_{\beta}^{\alpha} \varepsilon_{\gamma\delta} + (\gamma^3)_{\beta}^{\alpha} (\gamma^3)_{\gamma\delta} \right] E^{\gamma} \wedge \bar{E}^{\delta} - \frac{i}{4r^2} (\gamma_3)_{\beta}^{\alpha} \epsilon_{ab} E^a \wedge E^b, \\ \bar{\mathcal{R}}_{\beta}^{\alpha} &= d\bar{\Omega}_{\beta}^{\alpha} = \frac{1}{2r} \left[-\delta_{\beta}^{\alpha} \varepsilon_{\gamma\delta} + (\gamma^3)_{\beta}^{\alpha} (\gamma^3)_{\gamma\delta} \right] E^{\gamma} \wedge \bar{E}^{\delta} - \frac{i}{4r^2} (\gamma_3)_{\beta}^{\alpha} \epsilon_{ab} E^a \wedge E^b.\end{aligned}\tag{2.22}$$

These equations can be rewritten in a compact form

$$\mathcal{R} = \frac{i}{2r^2} M \epsilon_{ab} E^a \wedge E^b - \left(\frac{1}{2r} R \varepsilon_{\alpha\beta} + \frac{1}{r} M \gamma_{\alpha\beta}^3 \right) E^{\alpha} \wedge \bar{E}^{\beta},\tag{2.23}$$

where we assume that the angular momentum operator M acts on the tangent space vectors v^a and spinors ψ^{α} according to the following rules

$$M v^a = -i\epsilon^{ab} v^b, \quad M \psi^{\alpha} = -\frac{1}{2} (\gamma_3)_{\beta}^{\alpha} \psi^{\beta}.\tag{2.24}$$

The R-symmetry generator R acts on a complex superfield Φ carrying the R-charge q as follows

$$R\Phi = -q\Phi, \quad R\bar{\Phi} = q\bar{\Phi}.\tag{2.25}$$

2.4 Algebra of covariant derivatives

Let us consider covariant derivatives on the supercoset $\frac{\text{SU}(2|1)}{\text{U}(1) \times \text{U}(1)}$

$$\mathcal{D}_A = E_A + \Omega_A = (\mathcal{D}_a, \mathcal{D}_{\alpha}, \bar{\mathcal{D}}_{\alpha})\tag{2.26}$$

appearing in the decomposition of the covariant differential \mathcal{D}

$$\mathcal{D} = d + \Omega = E^A \mathcal{D}_A = E^a \mathcal{D}_a + E^{\alpha} \mathcal{D}_{\alpha} + \bar{E}^{\alpha} \bar{\mathcal{D}}_{\alpha}.\tag{2.27}$$

Using the fact that the covariant differential squares to the curvature $\mathcal{D}^2 = \mathcal{R}$, one gets the following relation for the covariant derivatives

$$T^A \mathcal{D}_A - E^A \wedge E^B \mathcal{D}_B \mathcal{D}_A = \mathcal{R}.\tag{2.28}$$

With the use of the explicit expressions for the supertorsion (2.19) and curvature (2.23) we find the (anti)commutation relations between the covariant derivatives on $\frac{\text{SU}(2|1)}{\text{U}(1) \times \text{U}(1)}$

$$\begin{aligned} [\mathcal{D}_a, \mathcal{D}_b] &= \frac{i}{r^2} \epsilon_{ab} M, \quad [\mathcal{D}_a, \mathcal{D}_\alpha] = -\frac{i}{2r} (\gamma_a)^\beta_\alpha \mathcal{D}_\beta, \quad [\mathcal{D}_a, \bar{\mathcal{D}}_\alpha] = -\frac{i}{2r} (\gamma_a)^\beta_\alpha \bar{\mathcal{D}}_\beta, \\ \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} &= i \gamma_{\alpha\beta}^a \mathcal{D}_a + \frac{1}{r} \gamma_{\alpha\beta}^3 M + \frac{1}{2r} \epsilon_{\alpha\beta} R, \\ \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} &= \{\bar{\mathcal{D}}_\alpha, \bar{\mathcal{D}}_\beta\} = 0. \end{aligned} \quad (2.29)$$

The generators M and R act on \mathcal{D}_A as follows

$$\begin{aligned} [M, \mathcal{D}_a] &= -i \epsilon_{ab} \mathcal{D}_b, \quad [M, \mathcal{D}_\alpha] = \frac{1}{2} (\gamma_3)^\beta_\alpha \mathcal{D}_\beta, \quad [M, \bar{\mathcal{D}}_\alpha] = \frac{1}{2} (\gamma_3)^\beta_\alpha \bar{\mathcal{D}}_\beta, \\ [R, \mathcal{D}_\alpha] &= \mathcal{D}_\alpha, \quad [R, \bar{\mathcal{D}}_\alpha] = -\bar{\mathcal{D}}_\alpha. \end{aligned} \quad (2.30)$$

In the coordinate system corresponding to the coset representative (2.6) the covariant derivatives have the following form

$$\begin{aligned} \mathcal{D}_a &= \partial_a + \left(\frac{i}{2r} (\gamma_a)^\alpha_\beta + \frac{i}{2} \omega_a (\gamma_3)^\alpha_\beta \right) \left(\theta^\beta \partial_\alpha + \bar{\theta}^\beta \bar{\partial}_\alpha \right) + i \omega_a M, \\ \mathcal{D}_\alpha &= \partial_\alpha + i \gamma_{\alpha\beta}^a \bar{\theta}^\beta \partial_a + \left[\frac{1}{2r} (\gamma^a)^\beta_\gamma (\gamma^a)_{\alpha\delta} + \frac{1}{2} (\gamma_3)^\beta_\gamma (\gamma^a)_{\alpha\delta} \omega_a \right] \theta^\gamma \bar{\theta}^\delta \partial_\beta + \frac{i}{4} \omega_a \epsilon^{ab} (\gamma^b)_\alpha^\beta \bar{\theta}^2 \bar{\partial}_\beta \\ &\quad - \bar{\theta}^\beta \left(\gamma_{\alpha\beta}^a \omega_a - \frac{1}{r} \gamma_{\alpha\beta}^3 \right) M + \frac{1}{2r} \bar{\theta}_\alpha R, \\ \bar{\mathcal{D}}_\alpha &= \bar{\partial}_\alpha. \end{aligned} \quad (2.31)$$

Here we used the explicit expressions for the superconnection given in (2.9) and the inverse supervielbein (2.13). One can check that the derivatives (2.31) obey the algebra (2.29).

Note that the derivative $\bar{\mathcal{D}}_\alpha = \bar{\partial}_\alpha$ is short in the coordinates corresponding to the coset representative (2.6). Therefore we refer to this coordinate system as the chiral basis. In principle, one can consider other coordinates, e.g., anti-chiral in which the derivative \mathcal{D}_α becomes short or real coordinates in which the both covariant spinor derivatives have a symmetric form.

2.5 Killing supervector

The $\text{SU}(2|1)$ transformations of a superfield $V(z)$ on $\frac{\text{SU}(2|1)}{\text{U}(1) \times \text{U}(1)}$

$$\delta V = \mathbb{K} V \quad (2.32)$$

are generated by the operator \mathbb{K} constructed with the use of the Killing supervector $\xi^A(z) = (\xi^a, \xi^\alpha, \bar{\xi}^\alpha)$,

$$\mathbb{K} = \xi^a \mathcal{D}_a + \xi^\alpha \mathcal{D}_\alpha + \bar{\xi}^\alpha \bar{\mathcal{D}}_\alpha - i \mu(z) M - i \rho(z) R. \quad (2.33)$$

Here $\mu(z)$ and $\rho(z)$ are local superfield parameters which are related to the components of the Killing supervector ξ^A such that \mathbb{K} commutes with all the covariant derivatives

$$[\mathbb{K}, \mathcal{D}_A] = 0. \quad (2.34)$$

Equation (2.34) implies a number of differential equations on the components of the Killing supervector and the superfunctions μ and ρ

$$[\mathcal{D}_a, \mathbb{K}] = 0 \Rightarrow$$

$$\mathcal{D}_{(a}\xi_{b)} = 0, \quad (2.35a)$$

$$\mathcal{D}_a \xi_\alpha = \frac{i}{2r}(\gamma_a)_{\alpha\beta} \xi^\beta, \quad \mathcal{D}_a \bar{\xi}_\alpha = \frac{i}{2r}(\gamma_a)_{\alpha\beta} \bar{\xi}^\beta, \quad (2.35b)$$

$$\mathcal{D}_a \mu = \frac{1}{r^2} \epsilon_{ab} \xi^b, \quad \mathcal{D}_{[a} \xi_{b]} = -\mu \epsilon_{ab}, \quad (2.35c)$$

$$\mathcal{D}_a \rho = 0. \quad (2.35d)$$

$$[\mathcal{D}_\alpha, \mathbb{K}] = 0 \Rightarrow$$

$$\mathcal{D}_\alpha \xi^a = i\gamma_{\alpha\beta}^a \bar{\xi}^\beta, \quad \mathcal{D}_\alpha \bar{\xi}^\beta = 0, \quad (2.36a)$$

$$\mathcal{D}_{(\alpha} \xi_{\beta)} + \frac{i}{2r} \xi^a \gamma_{\alpha\beta}^a + \frac{i}{2} \mu \gamma_{\alpha\beta}^3 = 0, \quad (2.36b)$$

$$\mathcal{D}_\alpha \xi^\alpha + 2i\rho = 0, \quad (2.36c)$$

$$\mathcal{D}_\alpha \mu - \frac{i}{r} \gamma_{\alpha\beta}^3 \bar{\xi}^\beta = 0, \quad (2.36d)$$

$$\mathcal{D}_\alpha \rho - \frac{i}{2r} \bar{\xi}_\alpha = 0. \quad (2.36e)$$

$$[\bar{\mathcal{D}}_\alpha, \mathbb{K}] = 0 \Rightarrow$$

$$\bar{\mathcal{D}}_\alpha \xi^a = i\gamma_{\alpha\beta}^a \xi^\beta, \quad \bar{\mathcal{D}}_\alpha \xi^\beta = 0, \quad (2.37a)$$

$$\bar{\mathcal{D}}_{(\alpha} \bar{\xi}_{\beta)} + \frac{i}{2r} \xi^a \gamma_{\alpha\beta}^a + \frac{i}{2} \mu \gamma_{\alpha\beta}^3 = 0, \quad (2.37b)$$

$$\bar{\mathcal{D}}_\alpha \bar{\xi}^\alpha - 2i\rho = 0, \quad (2.37c)$$

$$\bar{\mathcal{D}}_\alpha \mu - \frac{i}{r} \gamma_{\alpha\beta}^3 \xi^\beta = 0, \quad (2.37d)$$

$$\bar{\mathcal{D}}_\alpha \rho + \frac{i}{2r} \xi_\alpha = 0. \quad (2.37e)$$

In particular, (2.35a) and (2.35b) are the Killing vector and Killing spinor equations, respectively. Eqs. (2.36a) and (2.37a) show that the Killing spinor ξ^α is chiral while $\bar{\xi}^\alpha$ is antichiral and they are expressed in terms of covariant spinor derivatives of the Killing vector. The other equations allow one to express the superfunctions $\mu(z)$ and $\rho(z)$ in terms of the Killing vector and spinors. Thus, the equations (2.35)–(2.37) completely define the components of the Killing supervector and the functions μ and ρ in (2.33).

The general solution of the equations (2.35)–(2.37) has the following form

$$\begin{aligned} \xi^\alpha &= \bar{\mathcal{D}}^2 \mathcal{D}^\alpha \zeta, & \bar{\xi}^\alpha &= -\mathcal{D}^2 \bar{\mathcal{D}}^\alpha \zeta, & \xi^a &= -2i\gamma_{\alpha\beta}^a \bar{\mathcal{D}}^\alpha \mathcal{D}^\beta \zeta, \\ \mu &= -\frac{2i}{r} \gamma_{\alpha\beta}^3 \bar{\mathcal{D}}^\alpha \mathcal{D}^\beta \zeta, & \rho &= \frac{i}{2} \bar{\mathcal{D}}^\alpha \mathcal{D}_\alpha \zeta, \end{aligned} \quad (2.38)$$

where $\zeta = \zeta(x, \theta, \bar{\theta})$ is a covariantly constant superfield parameter with zero R-charge defined modulo gauge transformations,

$$\mathcal{D}_a \zeta = 0, \quad R\zeta = 0, \quad \zeta \sim \zeta - i\Lambda + i\bar{\Lambda}. \quad (2.39)$$

Here Λ is a chiral and covariantly constant superfunction, $\bar{\mathcal{D}}_\alpha \Lambda = 0$, $\mathcal{D}_a \Lambda = 0$. Using the properties (2.39) one can check that the superfields (2.38) solve for (2.35)–(2.37) and the superfield parameter ζ has the number of independent components which are in one-to-one correspondence with the parameters of the $SU(2|1) \times U(1)_A$ group where $U(1)_A$ is the group of external automorphisms of $SU(2|1)$.

As an example, let us consider a chiral superfield Φ , $\bar{\mathcal{D}}_\alpha \Phi = 0$. With the use of (2.38) its $SU(2|1)$ transformation can be represented in the following simple form

$$\begin{aligned} \delta\Phi &= \mathbb{K}\Phi = (\zeta^a \mathcal{D}_a + \zeta^\alpha \mathcal{D}_\alpha - i\mu M - i\rho R)\Phi \\ &= \bar{\mathcal{D}}^2 [(\mathcal{D}^\alpha \zeta)(\mathcal{D}_\alpha \Phi)]. \end{aligned} \quad (2.40)$$

This formula will be useful in the next sections.

3 Superfield actions with (2, 2) supersymmetry

The general form of the action for a superfield theory on the supercoset $\frac{SU(2|1)}{U(1) \times U(1)}$ is

$$S = \int d^6 z E \mathcal{L}_f + \int d^4 z \mathcal{E} \mathcal{L}_c + \int d^4 \bar{z} \bar{\mathcal{E}} \bar{\mathcal{L}}_c, \quad (3.1)$$

where \mathcal{L}_f and \mathcal{L}_c are full and chiral superspace Lagrangians, respectively. The full superspace measure $d^6 z E = d^2 x d^2 \theta d^2 \bar{\theta} E$ and the chiral one $d^4 z \mathcal{E} = d^2 x d^2 \theta \mathcal{E}$ are related to each other as follows

$$d^6 z E = -\frac{1}{4} d^4 z \mathcal{E} \bar{\mathcal{D}}^2. \quad (3.2)$$

In this section we will construct classical actions of the form (3.1) for gauge and matter superfields on the supercoset $\frac{SU(2|1)}{U(1) \times U(1)}$.

3.1 Gauge superfield

To describe a gauge theory on the supercoset $\frac{SU(2|1)}{U(1) \times U(1)}$ we extend the covariant derivatives \mathcal{D}_A with gauge superfield connections V_A which take values in the Lie algebra of a gauge group \mathcal{G} ,

$$\nabla_A = \mathcal{D}_A + V_A, \quad V_A = (V_a, V_\alpha, \bar{V}_\alpha). \quad (3.3)$$

Gauge superfield constraints are imposed by requiring that the gauge-covariant derivatives obey the commutation relations which correspond to the following deformation of the

algebra (2.29)

$$\begin{aligned}
 \{\nabla_\alpha, \nabla_\beta\} &= \{\bar{\nabla}_\alpha, \bar{\nabla}_\beta\} = 0, \\
 \{\nabla_\alpha, \bar{\nabla}_\beta\} &= i\gamma_{\alpha\beta}^a \nabla_a + \frac{1}{r} \gamma_{\alpha\beta}^3 M + \frac{1}{2r} \varepsilon_{\alpha\beta} R + i\varepsilon_{\alpha\beta} G + \gamma_{\alpha\beta}^3 H, \\
 [\nabla_a, \nabla_b] &= \frac{i}{r^2} \varepsilon_{ab} M + i\mathbf{F}_{ab}, \\
 [\nabla_a, \nabla_\alpha] &= -\frac{i}{2r} (\gamma_a)^\beta_\alpha \nabla_\beta - (\gamma_a)^\beta_\alpha \bar{W}_\beta, \quad [\nabla_a, \bar{\nabla}_\alpha] = -\frac{i}{2r} (\gamma_a)^\beta_\alpha \bar{\nabla}_\beta + (\gamma_a)^\beta_\alpha W_\beta, \\
 [R, \nabla_\alpha] &= \nabla_\alpha, \quad [R, \bar{\nabla}_\alpha] = -\bar{\nabla}_\alpha, \\
 [M, \nabla_\alpha] &= \frac{1}{2} (\gamma^3)^\beta_\alpha \nabla_\beta, \quad [M, \bar{\nabla}_\alpha] = \frac{1}{2} (\gamma^3)^\beta_\alpha \bar{\nabla}_\beta.
 \end{aligned} \tag{3.4}$$

Here G , H , W_α , \bar{W}_α and \mathbf{F}_{ab} are gauge superfield strengths subject to Bianchi identities. In particular, W_α is covariantly chiral while \bar{W}_α is covariantly anti-chiral,

$$\bar{\nabla}_\alpha W_\beta = 0, \quad \nabla_\alpha \bar{W}_\beta = 0. \tag{3.5}$$

They satisfy the ‘standard’ Bianchi identity,

$$\nabla^\alpha W_\alpha = \bar{\nabla}^\alpha \bar{W}_\alpha. \tag{3.6}$$

The spinorial superfield strengths W_α and \bar{W}_α are expressed in terms of the scalar superfield strengths G and H

$$\bar{W}_\alpha = \nabla_\alpha G = -i (\gamma^3)^\beta_\alpha \nabla_\beta H, \quad W_\alpha = \bar{\nabla}_\alpha G = i (\gamma^3)^\beta_\alpha \bar{\nabla}_\beta H. \tag{3.7}$$

G and H are linear superfields

$$\nabla^2 G = \bar{\nabla}^2 G = 0, \quad \nabla^2 H = \bar{\nabla}^2 H = 0. \tag{3.8}$$

Let us introduce the gauge potential V as

$$\nabla_\alpha = e^{-V} \mathcal{D}_\alpha e^V, \quad \bar{\nabla}_\alpha = \bar{\mathcal{D}}_\alpha. \tag{3.9}$$

The superfield strengths are expressed in terms of the gauge superfield V as follows

$$\begin{aligned}
 G &= \frac{i}{2} \bar{\mathcal{D}}^\alpha (e^{-V} \mathcal{D}_\alpha e^V), \quad H = -\frac{1}{2} \gamma_3^{\alpha\beta} \bar{\mathcal{D}}_\alpha (e^{-V} \mathcal{D}_\beta e^V), \\
 W_\alpha &= -\frac{i}{4} \bar{\mathcal{D}}^2 (e^{-V} \mathcal{D}_\alpha e^V), \quad \bar{W}_\alpha = \frac{i}{2} \nabla_\alpha \bar{\mathcal{D}}^\beta (e^{-V} \mathcal{D}_\beta e^V).
 \end{aligned} \tag{3.10}$$

The gauge transformation for V reads

$$e^V \longrightarrow e^{i\bar{\Lambda}} e^V e^{-i\Lambda}, \tag{3.11}$$

where Λ and $\bar{\Lambda}$ are (anti)chiral, $\bar{\mathcal{D}}_\alpha \Lambda = 0$, $\mathcal{D}_\alpha \bar{\Lambda} = 0$. The corresponding gauge transformations for the superfield strengths (3.10) are

$$G \rightarrow e^{i\Lambda} G e^{-i\Lambda}, \quad H \rightarrow e^{i\Lambda} H e^{-i\Lambda}, \quad W_\alpha \rightarrow e^{i\Lambda} W_\alpha e^{-i\Lambda}. \tag{3.12}$$

The super Yang-Mills action is given by the integral over the chiral superspace of the superfield strength W_α squared,

$$S_{\text{SYM}} = \frac{2}{g^2} \text{tr} \int d^4z \mathcal{E} W^\alpha W_\alpha, \quad (3.13)$$

where g is the gauge coupling constant of mass-dimension one, $[g] = 1$. Alternatively, using the identities (3.7) one can represent (3.13) as a full superspace action in the following two equivalent forms

$$S_{\text{SYM}} = -\frac{4}{g^2} \text{tr} \int d^6z E G^2 = -\frac{4}{g^2} \text{tr} \int d^6z E H^2. \quad (3.14)$$

The variation of the SYM action (3.13) or (3.14) with respect to the gauge potential V has the following form

$$\delta S_{\text{SYM}} = -\frac{4i}{g^2} \text{tr} \int d^6z E \Delta V \nabla^\alpha W_\alpha, \quad (3.15)$$

where $\Delta V = e^{-V} \delta e^V$ is the gauge-covariant variation.

The classical SYM action (3.14) is a particular case of a general action for the two superfield strengths G and H

$$\text{tr} \int d^6z E \mathcal{H}(G, H), \quad (3.16)$$

where \mathcal{H} is some function. The action of this form can appear as part of the low-energy effective action in two-dimensional gauge theories in the $\mathcal{N} = (2, 2)$ superspace. It would be interesting to find the explicit form of the function \mathcal{H} by direct quantum computations.

Although in supersymmetric two-dimensional gauge theories there is no Chern-Simons term, one can consider a model which can be obtained by dimensional reduction of the three-dimensional supersymmetric Chern-Simons theory to two dimensions. In terms of the gauge superfields introduced above this action has the form which is very similar to the $\mathcal{N} = 2$, $d = 3$ Chern-Simons action [30]

$$S_{\text{CS}} = i\kappa r \text{tr} \int_0^1 dt \int d^6z E \bar{\mathcal{D}}^\alpha (e^{-tV} \mathcal{D}_\alpha e^{tV}) e^{-tV} \partial_t e^{tV}, \quad (3.17)$$

where κ is a dimensionless coupling constant. This action has the non-local form because of the integration over the auxiliary parameter t , but its variation is local,

$$\delta S_{\text{CS}} = 2\kappa r \text{tr} \int d^6z E G \Delta V. \quad (3.18)$$

We stress that in contrast to the three-dimensional gauge theory, in two dimensions the action (3.17) is not topological, but describes the BF-type interaction of component fields (see eq. (3.31)). For gauge supermultiplet components this action was considered in [4].

In (3.17) the covariant spinor derivatives \mathcal{D}_α and $\bar{\mathcal{D}}_\beta$ are contracted with the $\varepsilon^{\alpha\beta}$ tensor, however, in two dimensions there is one more invariant tensor, namely $\gamma_{\alpha\beta}^3$, which can be used for the contraction of spinor indices. Hence, we can also consider the action

$$S_{\text{BF}} = \kappa r \text{tr} \int_0^1 dt \int d^6z E (\gamma^3)^{\alpha\beta} \bar{\mathcal{D}}_\alpha (e^{-tV} \mathcal{D}_\beta e^{tV}) e^{-tV} \partial_t e^{tV}, \quad (3.19)$$

which is supersymmetric and gauge invariant by the same reasoning as (3.17). The variation of the action (3.19) is also local

$$\delta S_{\text{BF}} = -2\kappa r \operatorname{tr} \int d^6 z E H \Delta V. \quad (3.20)$$

In terms of the component fields the action (3.19) was considered in [4].

Finally, we note that the Fayet-Iliopoulos term in the $\mathcal{N} = (2, 2)$ superspace under consideration has the standard form

$$S_{\text{FI}} = 4\xi \operatorname{tr} \int d^6 z E V, \quad (3.21)$$

where ξ is a dimensionless coupling constant.

3.1.1 Component structure

The $\mathcal{N} = (2, 2)$ vector multiplet consists of two scalars $\sigma(x)$ and $\eta(x)$, one vector $A_a(x) = -\frac{1}{2}\gamma_a^{\alpha\beta} A_{\alpha\beta}$, spinors $\lambda_\alpha(x)$ and $\bar{\lambda}_\alpha(x)$ and one auxiliary field $D(x)$. By dimensional reduction this supermultiplet is related to the $\mathcal{N} = 1$, $d = 4$ vector multiplet. In particular, the scalars σ and η originate from the (dimensionally reduced) components of a four-dimensional vector.

Let us now consider the component structure of the gauge superfield V . The unphysical components can be eliminated by imposing the Wess-Zumino gauge

$$V| = 0, \quad \mathcal{D}_\alpha V| = \bar{\mathcal{D}}_\alpha V| = 0, \quad \mathcal{D}^2 V| = \bar{\mathcal{D}}^2 V| = 0, \quad (3.22)$$

while the physical components appear in the following derivatives of the gauge superfield

$$\begin{aligned} \frac{1}{2} [\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta] V| &= A_{\alpha\beta} + \gamma_{\alpha\beta}^3 \eta + i\varepsilon_{\alpha\beta} \sigma, \\ -\frac{1}{2} \bar{\mathcal{D}}^2 \mathcal{D}_\alpha V| &= \lambda_\alpha, \quad -\frac{1}{2} \mathcal{D}^2 \bar{\mathcal{D}}_\alpha V| = \bar{\lambda}_\alpha, \\ \frac{1}{8} \{\mathcal{D}^2, \bar{\mathcal{D}}^2\} V| &= D, \end{aligned} \quad (3.23)$$

where $|$ denotes the component value of a superfield at $\theta = \bar{\theta} = 0$.

Using the relations (3.10) we find the component structure of the superfield strengths:

$$\begin{aligned} G| &= -\sigma, & H| &= -\eta, \\ W_\alpha| &= \frac{i}{2} \lambda_\alpha, & \bar{W}_\alpha| &= \frac{i}{2} \bar{\lambda}_\alpha, \\ \mathcal{D}^\alpha W_\alpha| &= -iD - \frac{\sigma}{r}, \\ \mathcal{D}_{(\alpha} W_{\beta)}| &= -\frac{i}{4} \gamma_{\alpha\beta}^3 \epsilon^{ab} F_{ab} - \frac{i}{2} \hat{\nabla}_{\alpha\beta} \sigma + \frac{i}{2} \epsilon^{ab} \gamma_{\alpha\beta}^b \hat{\nabla}_a \eta + \frac{1}{2} \gamma_{\alpha\beta}^3 [\eta, \sigma] - \frac{i}{2r} \gamma_{\alpha\beta}^3 \eta, \\ \mathcal{D}^2 W_\alpha| &= -\gamma_{\alpha\beta}^a \hat{\nabla}_a \bar{\lambda}^\beta + [\sigma, \bar{\lambda}_\alpha] - i\gamma_{\alpha\beta}^3 [\eta, \bar{\lambda}^\beta], \end{aligned} \quad (3.24)$$

where

$$\begin{aligned}\hat{\nabla}_a \bar{\lambda}^\beta &= \hat{\mathcal{D}}_a \bar{\lambda}^\beta + i [A_a, \bar{\lambda}^\beta], \\ \hat{\nabla}_a \sigma &= \hat{\mathcal{D}}_a \sigma + i [A_a, \sigma], \quad \left(\hat{\nabla}_{\alpha\beta} = \gamma_{\alpha\beta}^a \hat{\nabla}_a \right), \\ F_{ab} &= \hat{\mathcal{D}}_a A_b - \hat{\mathcal{D}}_b A_a + i [A_a, A_b]\end{aligned}\tag{3.25}$$

and $\hat{\mathcal{D}}_a = \partial_a + i\omega_a(x)M$ is a covariant derivative on S^2 .

Using the above relations we get the component structure of the $\mathcal{N} = (2, 2)$ SYM action (3.13)

$$\begin{aligned}S_{\text{SYM}} &= \frac{2}{g^2} \text{tr} \int d^6 z \mathcal{E} W^\alpha W_\alpha \\ &= -\frac{1}{g^2} \text{tr} \int d^2 x \sqrt{h} \left(W^\alpha \mathcal{D}^2 W_\alpha - \frac{1}{2} \mathcal{D}^\alpha W_\alpha \mathcal{D}^\beta W_\beta - \mathcal{D}_{(\alpha} W_{\beta)} \mathcal{D}^{(\alpha} W^{\beta)} \right) \Big|.\end{aligned}\tag{3.26}$$

Substituting (3.24) into (3.26) we find

$$\begin{aligned}S_{\text{SYM}} &= \frac{1}{2g^2} \text{tr} \int d^2 x \sqrt{h} \left[V_a V_a + V_3^2 + \left(iD + \frac{1}{r} \sigma \right)^2 \right. \\ &\quad \left. + i\lambda^\alpha \left(\gamma_{\alpha\beta}^a \hat{\nabla}_a \bar{\lambda}^\beta - [\sigma, \bar{\lambda}_\alpha] + i\gamma_{\alpha\beta}^3 [\eta, \bar{\lambda}^\beta] \right) \right],\end{aligned}\tag{3.27}$$

where

$$V_a = \hat{\nabla}_a \sigma + \epsilon_{ab} \hat{\nabla}_b \eta, \quad V_3 = \frac{1}{2} \epsilon^{ab} F_{ab} + i[\eta, \sigma] + \frac{1}{r} \eta.\tag{3.28}$$

Since, modulo a total derivative,

$$V_a V_a + V_3^2 = \left(\frac{1}{2} \epsilon^{ab} F_{ab} + \frac{1}{r} \eta \right)^2 + \hat{\nabla}_a \sigma \hat{\nabla}_a \sigma + \hat{\nabla}_a \eta \hat{\nabla}_a \eta - [\eta, \sigma]^2,\tag{3.29}$$

the action (3.27) takes the following equivalent form

$$\begin{aligned}S_{\text{SYM}} &= \frac{1}{2g^2} \text{tr} \int d^2 x \sqrt{h} \left[\left(\frac{1}{2} \epsilon^{ab} F_{ab} + \frac{1}{r} \eta \right)^2 + \hat{\nabla}_a \sigma \hat{\nabla}_a \sigma + \hat{\nabla}_a \eta \hat{\nabla}_a \eta - [\eta, \sigma]^2 \right. \\ &\quad \left. + \left(iD + \frac{1}{r} \sigma \right)^2 + i\lambda^\alpha \left(\gamma_{\alpha\beta}^a \hat{\nabla}_a \bar{\lambda}^\beta - [\sigma, \bar{\lambda}_\alpha] + i\gamma_{\alpha\beta}^3 [\eta, \bar{\lambda}^\beta] \right) \right].\end{aligned}\tag{3.30}$$

Similarly we find the component structure of the actions (3.17), (3.19) and (3.21):

$$S_{\text{CS}} = -\frac{i\kappa r}{4} \text{tr} \int d^2 x \sqrt{h} \left(\eta \epsilon^{ab} F_{ab} + \bar{\lambda}^\alpha \lambda_\alpha - 2i\sigma D + \frac{\eta^2}{r} - \frac{\sigma^2}{r} \right),\tag{3.31}$$

$$S_{\text{BF}} = \frac{\kappa r}{2} \text{tr} \int d^2 x \sqrt{h} \left(\eta D - \frac{i}{r} \eta \sigma + \frac{1}{2} (\gamma^3)^{\alpha\beta} \lambda_\alpha \bar{\lambda}_\beta - \frac{i}{2} \epsilon^{ab} F_{ab} \sigma \right),\tag{3.32}$$

$$S_{\text{FI}} = \xi \text{tr} \int d^2 x \sqrt{h} D.\tag{3.33}$$

The actions (3.31) and (3.32) were constructed in [4] for studying partition functions of supersymmetric gauge theories on S^2 . In this paper we gave the superfield forms (3.17) and (3.19) of these actions.

3.2 Chiral superfield

The dynamics on S^2 of a chiral superfield Φ ,

$$\bar{\mathcal{D}}_\alpha \Phi = 0, \quad \mathcal{D}_\alpha \bar{\Phi} = 0, \quad (3.34)$$

with the R-charge q ,

$$R\Phi = -q\Phi, \quad R\bar{\Phi} = q\bar{\Phi} \quad (3.35)$$

is described by the conventional Wess-Zumino action

$$S_{\text{WZ}} = 4 \int d^6 z E \bar{\Phi} \Phi + 2 \int d^4 z \mathcal{E} W(\Phi) + 2 \int d^4 \bar{z} \bar{\mathcal{E}} \bar{W}(\bar{\Phi}), \quad (3.36)$$

where $W(\Phi)$ is a chiral potential. Note that though the R-charge of the chiral superfield is arbitrary, the R-charge of the chiral potential is fixed $RW = -2W$ to have the opposite value of the R-charge of the chiral measure.

The chiral multiplet consists of a complex scalar ϕ , a spinor ψ_α and an auxiliary field F . These fields appear as the following components of the chiral superfield:

$$\begin{aligned} \phi(x) &= \Phi| & \bar{\phi}(x) &= \bar{\Phi}| \\ \psi_\alpha(x) &= \mathcal{D}_\alpha \Phi| & \bar{\psi}_\alpha(x) &= \bar{\mathcal{D}}_\alpha \bar{\Phi}| \\ F(x) &= -\frac{1}{2} \mathcal{D}^2 \Phi| & \bar{F}(x) &= -\frac{1}{2} \bar{\mathcal{D}}^2 \bar{\Phi}|. \end{aligned} \quad (3.37)$$

With such a definition of the component fields the Wess-Zumino action on S^2 has the conventional component form [4, 5]

$$\begin{aligned} S_{\text{WZ}} &= \int d^2 x \sqrt{h} \left(-\phi \hat{\mathcal{D}}^a \hat{\mathcal{D}}_a \bar{\phi} - \frac{q^2 - 2q}{4r^2} \phi \bar{\phi} - i \gamma_{\alpha\beta}^a \psi^\alpha \hat{\mathcal{D}}_a \bar{\psi}^\beta - \frac{q}{2r} \psi^\alpha \bar{\psi}_\alpha + F \bar{F} \right) \\ &+ \int d^2 x \sqrt{h} \left(W'(\phi) F - \frac{1}{2} W''(\phi) \psi^\alpha \psi_\alpha + \text{c.c.} \right). \end{aligned} \quad (3.38)$$

The interaction of the chiral superfield with the gauge superfield V in the adjoint representation is described by the action

$$S_{\text{ad}} = 4 \text{tr} \int d^6 z E e^{-V} \bar{\Phi} e^V \Phi. \quad (3.39)$$

It is straightforward to find the component structure of this action taking into account the definition of the component fields (3.23) and (3.37)

$$\begin{aligned} S_{\text{ad}} &= \text{tr} \int d^2 x \sqrt{h} \left[\hat{\nabla}^a \bar{\phi} \hat{\nabla}_a \phi - \frac{q^2 - 2q}{4r^2} \bar{\phi} \phi + D [\phi, \bar{\phi}] - \frac{iq}{r} \bar{\phi} [\sigma, \phi] \right. \\ &+ \bar{\phi} [\eta, [\eta, \phi]] + \bar{\phi} [\sigma, [\sigma, \phi]] + i \bar{\psi}^\alpha (\gamma^a)_\alpha^\beta \hat{\nabla}_a \psi_\beta - \frac{q}{2r} \bar{\psi}^\alpha \psi_\alpha \\ &\left. - \bar{\psi}^\alpha (\gamma_3)_\alpha^\beta [\eta, \psi_\beta] - i \bar{\psi}^\alpha [\sigma, \psi_\alpha] - [\bar{\phi}, \lambda^\alpha] \psi_\alpha - \bar{\psi}^\alpha [\bar{\lambda}_\alpha, \phi] + \bar{F} F \right]. \end{aligned} \quad (3.40)$$

The generalization to any other representation of the gauge group is straightforward.

4 Models with extended supersymmetry

In the previous section we considered supersymmetric field theories on S^2 with minimal $\mathcal{N} = (2, 2)$ supersymmetry. The supersymmetries (as well as other isometries of the coset $\frac{\text{SU}(2|1)}{\text{U}(1) \times \text{U}(1)})$ are generated by the operator \mathbb{K} given in (2.33).

In this section we consider field theories on S^2 with an extended number of supersymmetries using the $\mathcal{N} = (2, 2)$ superfield formulation. The examples to be discussed include the $\mathcal{N} = (4, 4)$ and $\mathcal{N} = (8, 8)$ SYM models, hypermultiplet and two-dimensional analogs of the Gaiotto-Witten [23] and ABJM [24] theories.

Note that the classical actions for field models with $\mathcal{N} = (4, 4)$ supersymmetry on S^2 can be, in principle, derived from [31–37] where the $\mathcal{N} = (4, 4)$ superfield supergravity with matter was studied using various approaches. Here we avoid the discussion of features of two-dimensional supergravity theories and construct the superfield actions with the use of algebraical methods.

In the $\mathcal{N} = (2, 2)$ superfield formulation the extra supersymmetries are realized as transformations that mix different superfields, e.g., the chiral and vector multiplets. Such transformations are associated with extra Killing spinors, say ϵ_α , which obey the equation

$$\hat{\mathcal{D}}_a \epsilon_\alpha = \frac{i}{2r} (\gamma_a)_\alpha^\beta \epsilon_\beta. \quad (4.1)$$

Here $\hat{\mathcal{D}}_a = \partial_a + i\omega_a(x)M$ is the covariant derivative on S^2 . The spinor ϵ_α appears as a component of a chiral ‘superfield’ parameter

$$\Upsilon = a + \theta^\alpha \epsilon_\alpha + \theta^2 b, \quad \bar{\mathcal{D}}_\alpha \Upsilon = 0, \quad (4.2)$$

subject to the covariant constancy condition

$$\mathcal{D}_a \Upsilon = 0. \quad (4.3)$$

Indeed, using the explicit form (2.31) of the derivative \mathcal{D}_a in chiral coordinates one can check that (4.3) implies (4.1) while the bosonic components a and b are constant, $\partial_a a = \partial_a b = 0$. These components should correspond to parameters of a R-symmetry group in a model with extended supersymmetry.

Recall that there is the sign ambiguity in the definition of the Killing spinors (4.1) such that a spinor $\tilde{\epsilon}_\alpha$ obeying the equation

$$\hat{\mathcal{D}}_a \tilde{\epsilon}_\alpha = -\frac{i}{2r} (\gamma_a)_\alpha^\beta \tilde{\epsilon}_\beta \quad (4.4)$$

is also a Killing spinor. As is pointed out in [18], the Killing spinors subject to the Killing spinor equations with different signs play important role in constructing field theories with extended supersymmetry on S^3 since they are independent in three dimensions. Field theories on S^3 which involve different numbers of “positive” and “negative” Killing spinors are, in general, not equivalent though they respect the same amount of supersymmetry. In two dimensions, however, such spinors are not independent. Indeed, given the spinor ϵ_α one can construct

$$\epsilon^L = \frac{1}{2} (\mathbf{1} + \gamma^3) \epsilon, \quad \epsilon^R = \frac{1}{2} (\mathbf{1} - \gamma^3) \epsilon, \quad (4.5)$$

such that

$$\epsilon = \epsilon^L + \epsilon^R, \quad \tilde{\epsilon} = \epsilon^L - \epsilon^R. \quad (4.6)$$

Thus, for the construction of supersymmetric field models on S^2 it is sufficient to consider only the Killing spinors ϵ_α obeying (4.1).

In what follows we will discuss in details the $\mathcal{N} = (4, 4)$ SYM theory on S^2 , while for other examples we will present only classical actions and corresponding extended supersymmetry transformations under which these actions are invariant.

4.1 $\mathcal{N} = (4, 4)$ SYM theory

To construct the action for the $\mathcal{N} = (4, 4)$ SYM theory on S^2 we will follow the same procedure as we used for the $\mathcal{N} = 4$ SYM model on S^3 in [18].

The $\mathcal{N} = (4, 4)$ gauge supermultiplet in the $\mathcal{N} = (2, 2)$ superspace is described by a gauge superfield $V(x, \theta, \bar{\theta})$ and a chiral superfield $\Phi(x, \theta, \bar{\theta})$ in the adjoint representation. The latter can have an arbitrary R-charge q

$$R\bar{\Phi} = q\bar{\Phi}, \quad R\Phi = -q\Phi. \quad (4.7)$$

A naive generalization of the flat space action for these superfields to the coset $\frac{\text{SU}(2|1)}{\text{U}(1) \times \text{U}(1)}$ is

$$S_0 = -\frac{4}{g^2} \text{tr} \int d^6 z E (G^2 - \bar{\Phi}\Phi), \quad (4.8)$$

where

$$\bar{\Phi} = e^{-V} \bar{\Phi} e^V, \quad \Phi = \Phi, \quad \nabla_\alpha \bar{\Phi} = 0, \quad \bar{\nabla}_\alpha \Phi = 0 \quad (4.9)$$

are covariantly (anti)chiral superfields. The action (4.8) is invariant under standard gauge transformations

$$\Delta V = i\bar{\Lambda} - i\Lambda, \quad \delta\Phi = i[\Lambda, \Phi], \quad \delta\bar{\Phi} = i[\bar{\Lambda}, \bar{\Phi}] \quad (4.10)$$

with the (anti)chiral superfield parameter Λ ($\bar{\Lambda}$).

We should find transformations of hidden $\mathcal{N} = (2, 2)$ supersymmetry which mix the superfields Φ and V . Such transformations are generated by the Killing spinors (4.1) which enter the chiral superfield parameter Υ given in (4.2) and subject to (4.3). Taking into account that such transformations should preserve covariant chirality of Φ and should close on the $\text{SU}(2)$ isometry of S^2 and an R-symmetry we find the following unique form of these transformations

$$\Delta_\Upsilon V = i(\Upsilon\bar{\Phi} - \bar{\Upsilon}\Phi), \quad \delta_\Upsilon \Phi = \bar{\nabla}^\alpha G \mathcal{D}_\alpha \Upsilon + \frac{q}{2r} G \Upsilon, \quad \delta_\Upsilon \bar{\Phi} = -\nabla^\alpha G \bar{\mathcal{D}}_\alpha \bar{\Upsilon} - \frac{q}{2r} G \bar{\Upsilon}, \quad (4.11)$$

where the R-charge of Υ should be the same as of Φ

$$R\bar{\Upsilon} = q\bar{\Upsilon}, \quad R\Upsilon = -q\Upsilon. \quad (4.12)$$

Indeed, using the algebra of the covariant derivatives (3.4) and the constraint (4.3) one can check that

$$\bar{\nabla}_\alpha \delta_\Upsilon \Phi = 0, \quad \nabla_\alpha \delta_\Upsilon \bar{\Phi} = 0. \quad (4.13)$$

The commutator of the two transformations (4.11) with superfield parameters Υ_1 and Υ_2 can be written as

$$\begin{aligned} [\delta_{\Upsilon_2}, \delta_{\Upsilon_1}] \Phi &= \bar{\nabla}^2 [(\mathcal{D}^\alpha \zeta)(\nabla_\alpha \Phi)], & [\delta_{\Upsilon_2}, \delta_{\Upsilon_1}] \bar{\Phi} &= -\nabla^2 [(\bar{\mathcal{D}}^\alpha \zeta)(\bar{\nabla}_\alpha \bar{\Phi})], \\ [\delta_{\Upsilon_2}, \delta_{\Upsilon_1}] G &= -2i\gamma_{\alpha\beta}^a \left(\bar{\mathcal{D}}^\alpha \mathcal{D}^\beta \zeta \right) \nabla_a G + (\bar{\mathcal{D}}^2 \mathcal{D}^\alpha \zeta) \nabla_\alpha G - (\mathcal{D}^2 \bar{\mathcal{D}}^\alpha \zeta) \bar{\nabla}_\alpha G, \end{aligned} \quad (4.14)$$

where

$$\zeta = \frac{1}{4} (\tilde{\Upsilon}_1 \Upsilon_2 - \tilde{\Upsilon}_2 \Upsilon_1). \quad (4.15)$$

The equations (4.14) show that the commutator of the two transformations (4.11) for the chiral superfield has exactly the form (2.40) while for the superfield strength G it has the general form (2.32), (2.33) with components of the Killing supervector given in (2.38). Therefore, the commutator of the transformations (4.11) closes on the symmetries of the coset $\frac{\text{SU}(2|1)}{\text{U}(1) \times \text{U}(1)}$ and, in particular, the hidden $\mathcal{N} = (2, 2)$ supersymmetry contained in (4.11) closes on the bosonic symmetries of the coset.

It is a simple exercise to check that the action (4.8) is not invariant under (4.11) unless $q = 0$,

$$\delta_\Upsilon S_0 = -\frac{2q}{rg^2} \text{tr} \int d^6 z E G (\tilde{\Upsilon} \Phi - \Upsilon \bar{\Phi}). \quad (4.16)$$

However, similar to the $\mathcal{N} = 4$ SYM model on S^3 [18], this term is canceled against the variation of the Chern-Simons-like action (3.17)

$$S_{\text{CS}} = -\frac{q}{rg^2} \text{tr} \int_0^1 dt \int d^6 z E \bar{\mathcal{D}}^\alpha (e^{-tV} \mathcal{D}_\alpha e^{tV}) e^{-tV} \partial_t e^{tV}, \quad (4.17)$$

$$\delta_\Upsilon S_{\text{CS}} = -\frac{2q}{rg^2} \text{tr} \int d^6 z E G (\Upsilon \bar{\Phi} - \tilde{\Upsilon} \Phi). \quad (4.18)$$

We thus find that for a generic q the classical action for the $\mathcal{N} = (4, 4)$ SYM on S^2 is

$$S_{\text{SYM}}^{\mathcal{N}=(4,4)} = -\frac{4}{g^2} \text{tr} \int d^6 z E \left[G^2 - \bar{\Phi} \Phi + \frac{q}{4r} \int_0^1 dt \bar{\mathcal{D}}^\alpha (e^{-tV} \mathcal{D}_\alpha e^{tV}) e^{-tV} \partial_t e^{tV} \right]. \quad (4.19)$$

Being manifestly invariant under $\text{SU}(2|1)$, this action is also invariant under the transformations (4.11), $\delta_\Upsilon S_{\text{SYM}}^{\mathcal{N}=(4,4)} = 0$. All together these transformations form the supergroup $\text{SU}(2|2) \times \text{SU}(2)_A$, where $\text{SU}(2)_A$ is the group of external automorphisms of $\text{SU}(2|2)$, which will manifest itself in the component form of the action.

4.1.1 Component structure of $\mathcal{N} = (4, 4)$ SYM on S^2

The classical action for the $\mathcal{N} = (4, 4)$ SYM theory on S^2 in terms of $\mathcal{N} = (2, 2)$ superfields is given by (4.19). We stress that this action is gauge invariant and $\mathcal{N} = (4, 4)$ supersymmetric for any value of q . It is interesting to consider the component structure of this action to find possible constraints on the parameter q .

The Lagrangian of the action (4.19) consist of three parts, namely, the pure $\mathcal{N} = (2, 2)$ SYM Lagrangian given by G^2 , the Lagrangian for the chiral superfield in the adjoint representation of the gauge group given by $\bar{\Phi} \Phi$ and, the Chern-Simons-like part given by the

last term in (4.19). The component structure of these three terms are given in (3.27), (3.40) and (3.31), respectively. Putting these expressions together we have

$$S_{\text{SYM}}^{\mathcal{N}=(4,4)} = \frac{1}{g^2} \text{tr} \int d^2x \sqrt{h} (\mathcal{L}_{\text{bos}} + \mathcal{L}_{\text{ferm}}), \quad (4.20)$$

$$\begin{aligned} \mathcal{L}_{\text{bos}} = & \frac{1}{8} \left(\epsilon^{ab} F_{ab} + \frac{q+2}{r} \eta \right)^2 + \frac{1}{2} \nabla^a \sigma \nabla_a \sigma + \frac{1}{2} \nabla^a \eta \nabla_a \eta + \nabla^a \bar{\phi} \nabla_a \phi \\ & + \frac{q(2-q)}{4r^2} \phi \bar{\phi} + \frac{q(2-q)}{8r^2} \sigma^2 - \frac{q(q+2)}{8r^2} \eta^2 + \bar{F} F - \frac{1}{2} (D')^2 \\ & + \frac{i(2-3q)}{2r} \sigma [\phi, \bar{\phi}] - \frac{1}{2} [\eta, \sigma]^2 + \frac{1}{2} [\phi, \bar{\phi}]^2 - [\sigma, \phi] [\sigma, \bar{\phi}] - [\eta, \phi] [\eta, \bar{\phi}], \end{aligned} \quad (4.21)$$

$$\begin{aligned} \mathcal{L}_{\text{ferm}} = & \frac{i}{2} \lambda^\alpha \left(\gamma_{\alpha\beta}^a \nabla_a \bar{\lambda}^\beta - [\sigma, \bar{\lambda}_\alpha] + i \gamma_{\alpha\beta}^3 [\eta, \bar{\lambda}^\beta] \right) + \frac{q}{4r} \bar{\lambda}^\alpha \lambda_\alpha \\ & + i \bar{\psi}^\alpha (\gamma^a)_\alpha^\beta \hat{\nabla}_a \psi_\beta - \frac{q}{2r} \bar{\psi}^\alpha \psi_\alpha - \bar{\psi}^\alpha (\gamma_3)_\alpha^\beta [\eta, \psi_\beta] \\ & - i \bar{\psi}^\alpha [\sigma, \psi_\alpha] - [\bar{\phi}, \lambda^\alpha] \psi_\alpha - \bar{\psi}^\alpha [\bar{\lambda}_\alpha, \phi], \end{aligned} \quad (4.22)$$

where we made the following shift of the auxiliary field D

$$D' = D + \frac{i(q-2)}{2r} \sigma - [\phi, \bar{\phi}]. \quad (4.23)$$

The scalars ϕ , $\bar{\phi}$ and σ and the auxiliary fields can be unified into $\text{SU}(2)_R$ and $\text{SU}(2)_A$ triplets, respectively,

$$\phi_I = (\phi_1, \phi_2, \phi_3), \quad F_A = (F_1, F_2, F_3), \quad (4.24)$$

where

$$\begin{aligned} \phi &= \frac{1}{\sqrt{2}} (\phi_1 - i\phi_2), \quad \bar{\phi} = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2), \quad \phi_3 = -\sigma, \\ F &= \frac{1}{\sqrt{2}} (F_1 - iF_2), \quad \bar{F} = \frac{1}{\sqrt{2}} (F_1 + iF_2), \quad F_3 = iD'. \end{aligned} \quad (4.25)$$

Note that the scalar η is an $\text{SU}(2)_R \times \text{SU}(2)_A$ singlet.

The spinor fields are unified into $\text{SU}(2)_R$ doublets $\psi_{i\alpha}$ ($i = 1, 2$)

$$\bar{\psi}^{1\alpha} = \frac{i}{\sqrt{2}} \bar{\lambda}^\alpha, \quad \psi_{1\alpha} = \frac{i}{\sqrt{2}} \lambda_\alpha, \quad \psi_{2\alpha} = \bar{\psi}_\alpha, \quad \bar{\psi}^{2\alpha} = \psi^\alpha. \quad (4.26)$$

Then, the Lagrangians (4.21) and (4.22) can be recast into the $\text{SU}(2)_R \times \text{SU}(2)_A$ covariant form

$$\begin{aligned} \mathcal{L}_{\text{bos}} = & \frac{1}{8} \left(\epsilon^{ab} F_{ab} + \frac{q+2}{r} \eta \right)^2 + \frac{1}{2} \hat{\nabla}^a \phi^I \hat{\nabla}_a \phi_I + \frac{1}{2} \hat{\nabla}^a \eta \hat{\nabla}_a \eta \\ & + \frac{q(2-q)}{8r^2} \phi^I \phi_I - \frac{q(q+2)}{8r^2} \eta^2 + \frac{1}{2} F^A F_A \\ & + \frac{2-3q}{12r} \epsilon^{IJK} \phi_I [\phi_J, \phi_K] - \frac{1}{2} [\eta, \phi^I] [\eta, \phi_I] - \frac{1}{4} [\phi^I, \phi^J] [\phi_I, \phi_J], \end{aligned} \quad (4.27)$$

$$\mathcal{L}_{\text{ferm}} = i \bar{\psi}^{i\alpha} (\gamma^a)_\alpha^\beta \hat{\nabla}_a \psi_{i\beta} - \frac{q}{2r} \bar{\psi}^{i\alpha} \psi_{i\alpha} - \bar{\psi}^{i\alpha} (\gamma_3)_\alpha^\beta [\eta, \psi_{i\beta}] + i \bar{\psi}^{i\alpha} (\gamma^I)_i^j [\phi_I, \psi_{j\alpha}], \quad (4.28)$$

where $(\gamma^I)_i^j$ are gamma-matrices corresponding to the $SU(2)_R$ group. Thus, we see that the action (4.19) being $\mathcal{N} = (4, 4)$ supersymmetric is invariant under $SU(2) \sim SO(3)$ isometry of S^2 and possesses $SU(2)_R \times SU(2)_A$ R-symmetry. These transformations form the supergroup $SU(2|2) \times SU(2)_A$ where $SU(2)_A$ acts as the group of external automorphisms of $SU(2|2)$.

Note that all the scalar fields in (4.27) have a non-negative mass squared only for

$$q = 0. \quad (4.29)$$

Therefore, though the action (4.19) is $\mathcal{N} = (4, 4)$ supersymmetric for any value of q , its zero value $q = 0$ is singled out among others by the requirement of the absence of tachyons in the theory. Recall that for the analogous $\mathcal{N} = 4$ SYM model on S^3 the constraint $q = 0$ appeared from somewhat different arguments, namely, that the $d = 3$ SYM action, containing a Chern-Simons term should be invariant under large gauge transformations [18].

It would be of interest to construct an analog of the action (4.19) in the AdS_2 space and to find constraints on the value of the R-charge q in that model. The $\mathcal{N} = 4$ SYM action in AdS_3 space in terms of $\mathcal{N} = 2$ superfields was considered in a recent paper [38].

4.2 Hypermultiplet

The hypermultiplet is described by a pair of chiral superfields (X_+, X_-) , $\bar{D}_\alpha X_\pm = 0$, which, in principle, can have different R-charges,

$$R\bar{X}_\pm = q_\pm \bar{X}_\pm, \quad RX_\pm = -q_\pm X_\pm. \quad (4.30)$$

The interaction of the hypermultiplet with the $\mathcal{N} = (4, 4)$ gauge multiplet (V, Φ) is described by the action

$$\begin{aligned} S_{\text{hyp}} = & 4 \operatorname{tr} \int d^6 z E (\bar{X}_+ e^V X_+ e^{-V} + \bar{X}_- e^V X_- e^{-V}) \\ & - 2\sqrt{2}i \operatorname{tr} \int d^4 z \mathcal{E} X_+ [\Phi, X_-] + 2\sqrt{2}i \operatorname{tr} \int d^4 \bar{z} \bar{\mathcal{E}} \bar{X}_+ [\bar{\Phi}, \bar{X}_-]. \end{aligned} \quad (4.31)$$

Here we consider the hypermultiplets in the adjoint representation of the gauge group although the generalization to any other representation is straightforward.

The chiral superfield Φ has an arbitrary R-charge q . However, in view of the presence of the chiral potential in the second line of (4.31) this charge is related to the R-charges of the hypermultiplet

$$q + q_+ + q_- = 2. \quad (4.32)$$

It is convenient to introduce covariantly (anti)chiral superfields

$$\bar{\mathcal{X}}_+ = e^{-V} \bar{X}_+ e^V, \quad \mathcal{X}_+ = X_+, \quad \bar{\mathcal{X}}_- = e^{-V} \bar{X}_- e^V, \quad \mathcal{X}_- = X_-. \quad (4.33)$$

For these superfields, the transformations of the hidden $\mathcal{N} = (2, 2)$ supersymmetry (which is parametrized by the Killing spinors ϵ_α entering the chiral superfield parameter Υ as in (4.2)) are

$$\delta \mathcal{X}_\pm = \pm \frac{1}{2\sqrt{2}} \bar{\nabla}^2 (\bar{\Upsilon} \bar{\mathcal{X}}_\mp), \quad \delta \bar{\mathcal{X}}_\pm = \mp \frac{1}{2\sqrt{2}} \nabla^2 (\Upsilon \mathcal{X}_\mp). \quad (4.34)$$

Under these transformations the action (4.31) varies as follows

$$\delta S_{\text{hyp}} = -\frac{q(2-q-2q_+)}{2\sqrt{2}r^2} \text{tr} \int d^4z \mathcal{E} \Upsilon \mathcal{X}_+ \mathcal{X}_- + \frac{q(2-q-2q_+)}{2\sqrt{2}r^2} \text{tr} \int d^4\bar{z} \bar{\mathcal{E}} \bar{\Upsilon} \bar{\mathcal{X}}_+ \bar{\mathcal{X}}_- . \quad (4.35)$$

This variation vanishes if one of the following conditions is satisfied

$$q_+ = q_- = 1 - \frac{q}{2}, \quad \text{or} \quad q = 0. \quad (4.36)$$

Note that for $q = 0$ the R-charges q_+ and q_- are not necessary equal to each other.

For $q \neq 0$ the R-charges q_+ and q_- are equal to each other and the chiral superfields X_+ and X_- form an SU(2) doublet

$$X_i = (X_+, X_-), \quad \bar{X}^i = (\bar{X}_+, \bar{X}_-). \quad (4.37)$$

In terms of these superfields the action (4.31) has the following compact form

$$S_{\text{hyp}} = 4 \text{tr} \int d^6z E \bar{\mathcal{X}}^i \mathcal{X}_i - \sqrt{2} i \text{tr} \int d^4z \mathcal{E} \mathcal{X}^i [\Phi, \mathcal{X}_i] + \sqrt{2} i \text{tr} \int d^4\bar{z} \bar{\mathcal{E}} \bar{\mathcal{X}}^i [\bar{\Phi}, \bar{\mathcal{X}}_i], \quad (4.38)$$

while the hidden supersymmetry transformations (4.34) simplify to

$$\delta \mathcal{X}_i = \frac{1}{2\sqrt{2}} \bar{\nabla}^2 (\bar{\Upsilon} \bar{\mathcal{X}}_i), \quad \delta \bar{\mathcal{X}}^i = \frac{1}{2\sqrt{2}} \nabla^2 (\Upsilon \mathcal{X}^i). \quad (4.39)$$

Here the SU(2) indices i, j are raised and lowered with the antisymmetric tensor ε_{ij} , $\varepsilon_{12} = \varepsilon^{21} = 1$.

4.3 $\mathcal{N} = (8, 8)$ SYM

The $\mathcal{N} = (8, 8)$ gauge multiplet consists of the $\mathcal{N} = (4, 4)$ vector multiplet (V, Φ) and a hypermultiplet (X_+, X_-) in the adjoint representation. The $\mathcal{N} = (8, 8)$ SYM action is described by the sum of the actions (4.19) and (4.38)

$$S_{\text{SYM}}^{\mathcal{N}=(8,8)} = S_{\text{SYM}}^{\mathcal{N}=(4,4)} + S_{\text{hyp}}. \quad (4.40)$$

Recall that q is the R-charge of the chiral superfield Φ while q_{\pm} are charges of the hypermultiplet related to q as $q_+ = q_- = 1 - \frac{q}{2}$. For arbitrary value of the charge q the action (4.40) has only $\mathcal{N} = (4, 4)$ supersymmetry. However, for $q = \frac{2}{3}$ the R-charges of all three chiral superfields coincide, $q_{\pm} = \frac{2}{3}$. In this case the three chiral superfields form an SU(3) triplet

$$\Phi_i = (\Phi, X_+, X_-), \quad \bar{\Phi}^i = (\bar{\Phi}, \bar{X}_+, \bar{X}_-), \quad R\bar{\Phi}^i = \frac{2}{3} \bar{\Phi}^i. \quad (4.41)$$

The action (4.40) can be recast into the following form

$$S = S_{\text{YM}} + S_{\text{CS}} + S_{\text{pot}}, \quad (4.42)$$

$$S_{\text{YM}} = -\frac{4}{g^2} \text{tr} \int d^6z E (G^2 - e^{-V} \bar{\Phi}^i e^V \Phi_i), \quad (4.43)$$

$$S_{\text{CS}} = -\frac{2}{3rg^2} \text{tr} \int_0^1 dt \int d^6z E \bar{\mathcal{D}}^\alpha (e^{-tV} \mathcal{D}_\alpha e^{tV}) e^{-tV} \partial_t e^{tV}, \quad (4.44)$$

$$S_{\text{pot}} = -\frac{i\sqrt{2}}{3g^2} \text{tr} \int d^4z \mathcal{E} \varepsilon^{ijk} \Phi_i [\Phi_j, \Phi_k] + \frac{i\sqrt{2}}{3g^2} \text{tr} \int d^4\bar{z} \bar{\mathcal{E}} \varepsilon_{ijk} \bar{\Phi}^i [\bar{\Phi}^j, \bar{\Phi}^k]. \quad (4.45)$$

One can check that this action is invariant under the following transformations of a hidden $\mathcal{N} = (6, 6)$ supersymmetry

$$\Delta V = i\Upsilon_i \bar{\Phi}^i - i\bar{\Upsilon}^i \Phi_i, \quad (4.46)$$

$$\delta \Phi_i = \bar{\nabla}^\alpha G \mathcal{D}_\alpha \Upsilon_i + \frac{1}{3r} G \Upsilon_i + \frac{1}{2\sqrt{2}} \varepsilon_{ijk} \bar{\nabla}^2 \left(\bar{\Upsilon}^j \bar{\Phi}^k \right), \quad (4.47)$$

$$\delta \bar{\Phi}^i = -\nabla^\alpha G \bar{\mathcal{D}}_\alpha \bar{\Upsilon}^i - \frac{1}{3r} G \bar{\Upsilon}^i - \frac{1}{2\sqrt{2}} \varepsilon^{ijk} \nabla^2 (\Upsilon_j \Phi_k), \quad (4.48)$$

where Φ_i and $\bar{\Phi}^i$ are covariantly (anti)chiral superfields which are defined similar to eq. (4.9) and Υ_i is a triplet of chiral superfield parameters, $\bar{\mathcal{D}}_\alpha \Upsilon_i$, subject to the constraint

$$\mathcal{D}_a \Upsilon_i = 0. \quad (4.49)$$

In components, the superfield parameter Υ_i contains the Killing spinors $\epsilon_{i\alpha}$ which, together with $\bar{\epsilon}_\alpha^i$ appearing in $\bar{\Upsilon}^i$, are responsible for the extra $\mathcal{N} = (6, 6)$ supersymmetry on S^2 . This supersymmetry extends the original manifest $\mathcal{N} = (2, 2)$ supersymmetry to $\mathcal{N} = (8, 8)$.

4.4 Gaiotto-Witten model reduced to S^2

In three dimensions the Gaiotto-Witten [23] and ABJM [24] models are superconformal theories with extended supersymmetry. They play an important role in the $\text{AdS}_4/\text{CFT}_3$ correspondence. The superfield action for the Gaiotto-Witten and ABJM models on S^3 were constructed in [18].

Being reduced to two dimensions, these theories are, of course, not superconformal, but still represent interesting two-dimensional supersymmetric models with extended supersymmetry. In particular, in a recent paper [39] a relation among the two-dimensional reduction of the ABJM theory and the q -deformed $\mathcal{N} = (4, 4)$ SYM models in flat space was studied. In this paper we consider analogous models on the two-sphere S^2 .

The Gaiotto-Witten theory is described by two gauge superfields V and \tilde{V} corresponding to two different gauge groups and by two chiral superfields (a hypermultiplet), X_+ and X_- , in the bi-fundamental representation. In general, the chiral superfields can have different R-charges

$$RX_\pm = -q_\pm X_\pm, \quad R\bar{X}_\pm = q_\pm \bar{X}_\pm. \quad (4.50)$$

We find that a two-dimensional counterpart of the Gaiotto-Witten action for these superfields has the following form

$$S_{\text{GW}} = S_{\text{CS}}[V] - S_{\text{CS}}[\tilde{V}] + S_X + S_{\text{FI}}, \quad (4.51)$$

$$S_X = 4 \text{tr} \int d^6 z E \left(\bar{X}_+ e^V X_+ e^{-\tilde{V}} + X_- e^{-V} \bar{X}_- e^{\tilde{V}} \right), \quad (4.52)$$

$$S_{\text{FI}} = \frac{i}{4} \kappa (q_+ - q_-) \text{tr} \int d^6 z E \left(V + \tilde{V} \right), \quad (4.53)$$

where the terms $S_{\text{CS}}[V]$ and $S_{\text{CS}}[\tilde{V}]$ have the form (3.17). This action is invariant under the following transformations

$$\begin{aligned} \Delta V &= \bar{\Sigma} \mathcal{X}_+ \mathcal{X}_- + \Sigma \bar{\mathcal{X}}_- \bar{\mathcal{X}}_+, & \Delta \tilde{V} &= \bar{\Sigma} \mathcal{X}_- \mathcal{X}_+ + \Sigma \bar{\mathcal{X}}_+ \bar{\mathcal{X}}_-, \\ \delta \mathcal{X}_\pm &= \pm \bar{\nabla}^2 (\bar{\Upsilon} \bar{\mathcal{X}}_\mp), & \delta \bar{\mathcal{X}}_\pm &= \pm \nabla^2 (\Upsilon \mathcal{X}_\mp), \end{aligned} \quad (4.54)$$

where \mathcal{X}_\pm and $\bar{\mathcal{X}}_\pm$ are covariantly (anti)chiral superfields,

$$\bar{\mathcal{X}}_+ = e^{-\tilde{V}} \bar{X}_+ e^V, \quad \mathcal{X}_+ = X_+, \quad \bar{\mathcal{X}}_- = e^{-V} \bar{X}_- e^{\tilde{V}}, \quad \mathcal{X}_- = X_-, \quad (4.55)$$

and Υ ($\bar{\Upsilon}$) are (anti)chiral superfield parameters subject to the constraint (4.3). They contain the Killing spinors ϵ_α and $\bar{\epsilon}_\alpha$ as their components. The superfield parameters Σ and $\bar{\Sigma}$ are not independent, but are related to Υ and $\bar{\Upsilon}$ as

$$\mathcal{D}_\alpha \Sigma = \frac{8i}{\kappa r} \bar{\mathcal{D}}_\alpha \bar{\Upsilon}, \quad \bar{\mathcal{D}}_\alpha \bar{\Sigma} = \frac{8i}{\kappa r} \mathcal{D}_\alpha \Upsilon. \quad (4.56)$$

These equations define Σ and $\bar{\Sigma}$ in terms of Υ and $\bar{\Upsilon}$ in the unique way. For instance, for the chiral superfield parameter Υ given in the form (4.2) we find the following component field decomposition for $\bar{\Sigma}$ in the chiral coordinate system

$$\begin{aligned} \bar{\Sigma} = \frac{8i}{\kappa r} & \left(\frac{q_+ + q_- - 2}{4r} \bar{\theta}^2 a + \bar{\theta}^\alpha \epsilon_\alpha + \frac{q_+ + q_- - 2}{4r} \bar{\theta}^2 \theta^\alpha \epsilon_\alpha \right. \\ & \left. + 2\theta^\alpha \bar{\theta}_\alpha b + \frac{q_+ + q_- - 2}{4r} \theta^2 \bar{\theta}^2 b - \frac{4r}{q_+ + q_-} b \right). \end{aligned} \quad (4.57)$$

Note that the R-charges of Υ and Σ are expressed in terms of q_\pm as follows

$$R\Upsilon = (q_+ + q_- - 2)\Upsilon, \quad R\Sigma = -(q_+ + q_-)\Sigma. \quad (4.58)$$

We point out that the FI-term in (4.51) drops out for $q_+ = q_-$. Effectively, it compensates the difference of the R-charges of the chiral superfields such that the action remains $\mathcal{N} = (4, 4)$ supersymmetric.

4.5 ABJ(M) theory reduced to S^2

ABJM theory is similar to the Gaiotto-Witten model. It is also described by two gauge superfields V and \tilde{V} , but it has two copies of chiral superfields in the bi-fundamental representation,

$$X_{+i}, \quad X_-^i, \quad i = 1, 2. \quad (4.59)$$

A priori, we assume that these superfields have arbitrary R-charges

$$RX_{+i} = -q_+ X_{+i}, \quad RX_-^i = -q_- X_-^i. \quad (4.60)$$

The transformations of the hidden $\mathcal{N} = (4, 4)$ supersymmetry are analogous to those for the ABJM model on S^3 [18]

$$\begin{aligned} \Delta V &= \frac{8i}{\kappa r} \left(\bar{\Upsilon}_j^i \mathcal{X}_{+i} \mathcal{X}_-^j + \Upsilon_i^j \bar{\mathcal{X}}_{-j} \bar{\mathcal{X}}_+^i \right), \\ \Delta \tilde{V} &= \frac{8i}{\kappa r} \left(\bar{\Upsilon}_i^j \mathcal{X}_-^i \mathcal{X}_{+j} + \Upsilon_i^j \bar{\mathcal{X}}_+^i \bar{\mathcal{X}}_{-j} \right), \end{aligned} \quad (4.61)$$

$$\delta \mathcal{X}_{+i} = \bar{\nabla}^2 \left(\bar{\Upsilon}_i^j \bar{\mathcal{X}}_{-j} \right), \quad \delta \mathcal{X}_-^j = -\bar{\nabla}^2 \left(\bar{\Upsilon}_i^j \bar{\mathcal{X}}_+^i \right), \quad (4.62)$$

$$\delta \bar{\mathcal{X}}_+^i = \nabla^2 \left(\Upsilon_i^j \mathcal{X}_{+j} \right), \quad \delta \bar{\mathcal{X}}_{-j} = -\nabla^2 \left(\Upsilon_i^j \mathcal{X}_{+i} \right), \quad (4.63)$$

where $\mathcal{X}_{\pm i}$ and $\bar{\mathcal{X}}_{\pm i}$ are covariantly (anti)chiral superfields defined similar to (4.55), and Υ^i_j is a quartet of chiral superfield parameters each of which is constrained by (4.3). The anti-chiral superfield parameters are now not independent. They are expressed in terms of Υ^i_j

$$\bar{\Upsilon}^i_j = r\mathcal{D}^2\Upsilon^i_j. \quad (4.64)$$

This equation restricts the number of independent parameters in $\bar{\Upsilon}^i_j$ and Υ^i_j such that they involve four Killing spinors $(\epsilon^i_j)_\alpha$ which, together with the manifest $\mathcal{N} = (2, 2)$ supersymmetry, form the $\mathcal{N} = (6, 6)$ supersymmetry of the ABJ(M) model reduced to S^2 .

The action invariant under (4.63) has the following form

$$S_{\text{ABJM}} = S_{\text{CS}}[V] - S_{\text{CS}}[\tilde{V}] + S_X + S_{\text{pot}} + S_{\text{FI}}, \quad (4.65)$$

$$S_X = 4\text{tr} \int d^6z E \left(\bar{X}_+^i e^V X_{+i} e^{-\tilde{V}} + X_-^i e^{-V} \bar{X}_{-i} e^{\tilde{V}} \right), \quad (4.66)$$

$$\begin{aligned} S_{\text{pot}} = & -\frac{4}{\kappa} \text{tr} \int d^4z \mathcal{E} \left(X_{+i} X_-^i X_{+j} X_-^j - X_-^i X_{+i} X_-^j X_{+j} \right) \\ & -\frac{4}{\kappa} \text{tr} \int d^4\bar{z} \bar{\mathcal{E}} \left(\bar{X}_{-i} \bar{X}_+^i \bar{X}_{-j} \bar{X}_+^j - \bar{X}_+^i \bar{X}_{-i} \bar{X}_+^j \bar{X}_{-j} \right), \end{aligned} \quad (4.67)$$

$$S_{\text{FI}} = \frac{i}{4} \kappa (q_+ - q_-) \text{tr} \int d^6z E \left(V + \tilde{V} \right). \quad (4.68)$$

The presence of the term S_{pot} imposes the constraint on the R-charges q_\pm

$$q_+ + q_- = 1. \quad (4.69)$$

Therefore, only one of them is independent.

Similarly to the Gaiotto-Witten model (4.51), the action (4.65) has the FI-term which effectively compensates the difference of the R-charges of the chiral superfields such that it respects the symmetry (4.63) for an arbitrary value of q_+ . Obviously, for $q_+ = q_- = \frac{1}{2}$ the FI-term drops out.

5 One-loop partition functions

One-loop partition functions in the $\mathcal{N} = (2, 2)$ gauge and matter models on S^2 were computed in [4, 5] using the component field approach. For supersymmetric field theories the partition functions are given by the ratio of determinants of operators of quadratic fluctuations of fermionic and bosonic fields. As a rule, there are many cancellations among contributions to these determinants due to supersymmetry, so the final result usually looks quite simple. As in the case of superfield models on S^3 considered in [18], the use of the superfield approach makes these cancellations automatic. In this section we re-derive the results of one-loop partition functions of the chiral and gauge $\mathcal{N} = (2, 2)$ multiplets on S^2 using the superfield methods.

5.1 Chiral superfield interacting with background gauge superfield

5.1.1 Single chiral superfield interacting with Abelian gauge superfield

Let us consider the model of a chiral superfield Φ minimally interacting with an Abelian gauge superfield V ,

$$S = 4 \int d^6 z E \bar{\Phi} e^V \Phi = 4 \int d^6 z E \bar{\Phi} \Phi, \quad (5.1)$$

where

$$\bar{\Phi} = \bar{\Phi} e^V, \quad \Phi = \Phi \quad (5.2)$$

are the covariantly (anti)chiral superfields. In the one-loop approximation the partition function Z is given by the exponent of the effective action Γ , $Z = e^\Gamma$. The latter is proportional to the trace of the logarithm of the second variational derivative of the classical action

$$\Gamma = -\frac{1}{2} \text{Tr} \ln S''. \quad (5.3)$$

In the model (5.1) it is more convenient to compute the variation of the effective action, $\delta\Gamma$, which is expressed in terms of the effective current $\langle J \rangle$ as follows

$$\delta\Gamma = \int d^6 z E \delta V \langle J \rangle. \quad (5.4)$$

The effective current $\langle J \rangle$, in its turn, is related to the Green's function of the chiral superfield $\langle \bar{\Phi}(z) \Phi(z') \rangle$ considered at coincident superspace points, $\langle \bar{\Phi} \Phi \rangle \equiv \langle \bar{\Phi}(z) \Phi(z') \rangle|_{z=z'}$,

$$\langle J \rangle = \left\langle \frac{\delta S}{\delta V} \right\rangle = 4 \langle \bar{\Phi} \Phi \rangle. \quad (5.5)$$

In what follows we denote this Green's function as $\langle \bar{\Phi}(z) \Phi(z') \rangle \equiv G_{-+}(z, z')$. It obeys the equation

$$\bar{\nabla}^2 G_{-+}(z, z') = \delta_+(z, z'), \quad (5.6)$$

where $\delta_+(z, z')$ is a chiral delta-function ($\bar{\nabla}_\alpha \delta_+(z, z') = 0$),

$$\delta_+(z, z') = -\frac{1}{4} \bar{\nabla}^2 \delta^6(z, z'), \quad \delta^6(z, z') = \frac{1}{E} \delta^2(x - x') \delta^2(\theta - \theta') \delta^2(\bar{\theta} - \bar{\theta}'). \quad (5.7)$$

As a result, to obtain the variation of the effective action (5.4) we should find the Green's function G_{-+} at coincident superspace points.

The procedure of computing Green's functions of chiral superfields in four-dimensional superspace was developed in [40, 41]. Following this procedure, we express G_{-+} in terms of the covariantly chiral Green's function G_+ ,

$$G_{-+}(z, z') = -\frac{1}{4} \nabla^2 G_+(z, z'), \quad (5.8)$$

where G_+ obeys

$$\square_+ G_+(z, z') = -\delta_+(z, z'), \quad \square_+ \equiv \frac{1}{4} \bar{\nabla}^2 \nabla^2. \quad (5.9)$$

Using the algebra of covariant derivatives (3.4) we find the explicit form of the operator \square_+ acting on a chiral superfield

$$\square_+ = -\nabla^a \nabla_a + \frac{1}{4r^2} + \left(H + \frac{1}{r}M\right)^2 + \left(G - \frac{i}{2r}(R+1)\right)^2 + i(\bar{\nabla}^\alpha \bar{W}_\alpha) + 2iW^\alpha \nabla_\alpha. \quad (5.10)$$

Let us take a very particular background gauge superfield $V = V_0$ such that its superfield strengths G and H are constant while W_α and \bar{W}_α vanish, namely,

$$G = -\sigma = \text{const}, \quad H = -\eta = \text{const}, \quad W_\alpha = \bar{W}_\alpha = 0, \quad (5.11)$$

where σ and η are the scalar fields in the $\mathcal{N} = (2, 2)$ gauge supermultiplet. Using the equations (3.24) one can show that this background corresponds to the following values of the component fields

$$\eta = -\frac{n}{2r} = \text{const}, \quad F_{12} = \frac{n}{2r^2}, \quad \lambda_\alpha = \bar{\lambda}_\alpha = 0, \quad \sigma = \sigma_0 = \text{const}, \quad D = \frac{i}{r}\sigma_0. \quad (5.12)$$

Here n is integer owing to the quantization of the gauge field flux $\frac{1}{2\pi} \int F = n \in \mathbb{Z}$, [42], while σ_0 is an arbitrary real number. As a result, this background is parametrized by the pair of the parameters (n, σ_0) which appear as arguments of the partition function $Z = Z(n, \sigma_0)$. Note that exactly this background for the $\mathcal{N} = (2, 2)$ gauge supermultiplet was considered in [4, 5] in the application of the localization method to supersymmetric models on S^2 .

For the background (5.11) the form of the operator (5.10) acting on the chiral superfields with R-charge q simplifies,

$$\square_+ = -\nabla^a \nabla_a + m^2, \quad m^2 \equiv G^2 + H^2 + \frac{i}{r}G(q-1) + \frac{q(2-q)}{4r^2}, \quad (5.13)$$

where m is the effective mass. Here ∇_a is the superspace derivative which includes the gauge field connection A_a with constant field strength $F_{12} = \frac{n}{2r^2}$. In purely bosonic case the operator $\nabla^a \nabla_a$ is usually referred to as the covariant Laplacian on S^2 with a monopole gauge field background [4, 5].

For the gauge superfield background described above the chiral Green's function G_+ (5.9) can be written as

$$G_+(z, z') = -\frac{1}{4}\bar{\nabla}^2 G_o(z, z') = -\frac{1}{4}\bar{\nabla}'^2 G_o(z, z'), \quad (5.14)$$

where $\bar{\nabla}'^2$ acts on z' and $G_o(z, z')$ solves

$$\square_o G_o(z, z') = -\delta^6(z, z'), \quad \square_o = -\nabla^a \nabla_a + m^2. \quad (5.15)$$

The operator \square_o has the same expression as \square_+ given in eq. (5.13), but it acts on the superfields defined in the full superspace rather than on the chiral superfields. To check that (5.14) obeys (5.9) one should use the identities

$$[\nabla^2, \square_o] = [\bar{\nabla}^2, \square_o] = 0, \quad (5.16)$$

which hold for the gauge superfield background under consideration.

Combining (5.8) with (5.14) we find

$$G_{-+}(z, z') = \frac{1}{16} \nabla^2 \bar{\nabla}'^2 G_o(z, z') = -\frac{1}{16} \nabla^2 \bar{\nabla}'^2 \frac{1}{-\nabla^a \nabla_a + m^2} \delta^6(z, z'). \quad (5.17)$$

Next, using (5.16) we commute the operators ∇^2 and $\bar{\nabla}'^2$ with $(-\nabla^a \nabla_a + m^2)^{-1}$ and consider the Green's function (5.17) at coincident superspace points

$$G_{-+}(z, z) = -\frac{1}{-\nabla^a \nabla_a + m^2} \frac{1}{16} \nabla^2 \bar{\nabla}'^2 \delta^6(z, z')|_{z=z'} = -\frac{1}{\Delta_{S^2} + m^2} \delta^2(x, x')|_{x=x'}. \quad (5.18)$$

Note that to get a non-vanishing result, all the fermionic components of the superspace delta-function $\delta^6(z, z')$ should be cancelled by the operators ∇^2 and $\bar{\nabla}'^2$. The remaining expression is nothing but the trace of the inverse of the purely bosonic Laplacian Δ_{S^2} acting on the scalar fields on the S^2 -sphere

$$-\text{tr} \frac{1}{\Delta_{S^2} + m^2} \propto -\sum_j \frac{d_j}{\lambda_j + m^2}, \quad (5.19)$$

where λ_j are the eigenvalues of the Laplace operator on S^2 in the monopole background and d_j are their degeneracies [4, 5]

$$\lambda_j = \frac{1}{r^2} j(j+1) - \frac{n^2}{4r^2}, \quad d_j = 2j+1, \quad j = \frac{|n|}{2}, \frac{|n|}{2} + 1, \frac{|n|}{2} + 2, \dots \quad (5.20)$$

The sum (5.19) is divergent. Regularizing it in a standard way, $\sum \frac{1}{n} = \zeta(1) = \gamma$, we find

$$\begin{aligned} G_{-+}(z, z) &= \frac{1}{4\pi} \left(\psi \left(\frac{1}{2} + \frac{|n|}{2} - \frac{1}{2} \sqrt{1 - 4m^2 r^2 + n^2} \right) + \psi \left(\frac{1}{2} + \frac{|n|}{2} + \frac{1}{2} \sqrt{1 - 4m^2 r^2 + n^2} \right) \right) \\ &= \frac{1}{4\pi} \left(\psi \left(\frac{q}{2} + \frac{|n|}{2} + ir\sigma_0 \right) + \psi \left(1 + \frac{|n|}{2} - \frac{q}{2} - ir\sigma_0 \right) \right), \end{aligned} \quad (5.21)$$

where $\psi(z)$ is the digamma function which is related to the Euler gamma function by $\psi(z) = \Gamma'(z)/\Gamma(z)$. Here we used the explicit expression for the effective mass squared m^2 given in (5.13) which implies the identity

$$\sqrt{1 - 4m^2 r^2 + n^2} = 1 - q - 2ir\sigma_0. \quad (5.22)$$

As a result, the effective current is

$$\langle J \rangle = \frac{1}{\pi} \left(\psi \left(\frac{q}{2} + \frac{|n|}{2} + ir\sigma_0 \right) + \psi \left(1 + \frac{|n|}{2} - \frac{q}{2} - ir\sigma_0 \right) \right). \quad (5.23)$$

Now we substitute this effective current into the variation of the effective action (5.4) and perform integrations over the Grassmann and bosonic superspace coordinates,

$$\begin{aligned} \delta\Gamma &= \int d^6z E \langle J \rangle \delta V = \frac{1}{4} \int d^2x \sqrt{h} \langle J \rangle \delta D = \frac{i}{4r} \int d^2x \sqrt{h} \langle J \rangle \delta\sigma_0 \\ &= \frac{i}{4\pi r} \int d^2x \sqrt{h} \delta\sigma_0 \left(\psi \left(\frac{q}{2} + \frac{|n|}{2} + ir\sigma_0 \right) + \psi \left(1 + \frac{|n|}{2} - \frac{q}{2} - ir\sigma_0 \right) \right) \\ &= ir\delta\sigma_0 \left(\psi \left(\frac{q}{2} + \frac{|n|}{2} + ir\sigma_0 \right) + \psi \left(1 + \frac{|n|}{2} - \frac{q}{2} - ir\sigma_0 \right) \right). \end{aligned} \quad (5.24)$$

The integration over the Grassmann variables in the first line of (5.24) is similar to the computation of the component form of the FI-term (3.33) from the superfield action (3.21). Here we also used the relation between the values of the auxiliary field D and the scalar σ for the considered background (5.12). When passing from the second to the third line in (5.24) we used the fact that the integrand is independent of x and the remaining integration is just the volume of S^2 , $\int d^2x \sqrt{h} = \text{Vol}(S^2) = 4\pi r^2$.

It is a simple exercise to restore the effective action from its variation (5.24)

$$\Gamma = \ln \frac{\Gamma\left(\frac{q}{2} + \frac{|n|}{2} + ir\sigma_0\right)}{\Gamma\left(1 + \frac{|n|}{2} - \frac{q}{2} - ir\sigma_0\right)}. \quad (5.25)$$

Thus, the partition function of the chiral multiplet on the background (5.12) is

$$Z = e^\Gamma = \frac{\Gamma\left(\frac{q}{2} + \frac{|n|}{2} + ir\sigma_0\right)}{\Gamma\left(1 + \frac{|n|}{2} - \frac{q}{2} - ir\sigma_0\right)}. \quad (5.26)$$

This partition function was originally computed in [4, 5] using the component field formulation of the model (5.1). Note that the component field computations involve the spectra of both the (bosonic) Laplacian and the (fermionic) Dirac operator on S^2 , but most of the eigenvalues of these operators cancel against each other in the ratio of the one-loop determinants of quadratic fluctuations of the bosonic and fermionic modes. In superspace, we obtained the same result (5.26) without explicit use of the fermionic spectrum, only the knowledge of the purely bosonic spectrum (5.20) was necessary. With the use of the superfield Green's functions of the chiral superfields all cancellations among bosons and fermions become automatic.

5.1.2 Chiral superfield in adjoint representation

Consider the model of a chiral superfield Φ interacting with a background non-Abelian gauge superfield V in the adjoint representation (3.39). We assume that the gauge group is $U(N)$ and the background gauge superfield takes values in the Cartan subalgebra,

$$V = \text{diag}(V_1, V_2, \dots, V_N), \quad (5.27)$$

where each of the diagonal elements V_I in (5.27) has constant superfield strengths,

$$\begin{aligned} W_{I\alpha} &= \bar{W}_{I\alpha} = 0, \\ G_I &= \frac{i}{2} \bar{\mathcal{D}}^\alpha \mathcal{D}_\alpha V_I = -\sigma_I = \text{const}, \\ H_I &= -\frac{1}{2} (\gamma^3)^{\alpha\beta} \bar{\mathcal{D}}_\alpha \mathcal{D}_\beta V_I = -\eta_I = \text{const}. \end{aligned} \quad (5.28)$$

In components, such a background is given by (5.12), but now we will have a set of N independent pairs (n_I, σ_I) as arguments of the partition function, $Z = Z(n_I, \sigma_I)$.

The (anti)chiral superfield Φ ($\bar{\Phi}$) is a matrix in the $u(N)$ Lie algebra. It can be expanded in the basis elements $e_{IJ} = (e_{IJ})_{KL} = \delta_{IK}\delta_{JL}$

$$\Phi = \sum_{I \neq J}^N e_{IJ} \Phi_{IJ} + \sum_{I=1}^N e_{II} \Phi_I, \quad \bar{\Phi} = \sum_{I \neq J}^N e_{JI} \bar{\Phi}_{IJ} + \sum_{I=1}^N e_{II} \bar{\Phi}_I. \quad (5.29)$$

Note that the superfields Φ_I and $\bar{\Phi}_I$ in (5.29) correspond to the diagonal elements of the $u(N)$ matrix. These elements do not interact with the background gauge superfield (5.27) and we omit them in what follows. The off-diagonal elements Φ_{IJ} enter the action (3.39) as follows

$$S_{\text{ad}} = \sum_{I \neq J}^N \int d^6 z E \bar{\Phi}_{IJ} \Phi_{IJ}, \quad (5.30)$$

where Φ_{IJ} are chiral $\bar{D}_\alpha \Phi_{IJ} = 0$ while $\bar{\Phi}_{IJ}$ are covariantly antichiral,

$$e^{V_J - V_I} \mathcal{D}_\alpha e^{V_I - V_J} \bar{\Phi}_{IJ} = 0 \quad \text{for } I < J, \quad e^{V_I - V_J} \mathcal{D}_\alpha e^{V_J - V_I} \bar{\Phi}_{IJ} = 0 \quad \text{for } I > J. \quad (5.31)$$

Each element in the sum (5.30) has the form (5.1). Hence, the partition function in the model (3.39) is given by the product of the expressions (5.26)

$$Z_\Phi = \prod_{I \neq J}^N \frac{\Gamma\left(\frac{q}{2} + \frac{|n_I - n_J|}{2} + ir(\sigma_I - \sigma_J)\right)}{\Gamma\left(1 - \frac{q}{2} + \frac{|n_I - n_J|}{2} - ir(\sigma_I - \sigma_J)\right)}. \quad (5.32)$$

In a similar way one can find the partition function of the chiral superfield in an arbitrary representation of the gauge group.

Note that for $q = 1$ this partition function trivializes,

$$Z_\Phi|_{q=1} = 1. \quad (5.33)$$

This property is similar to the one of the partition function of the chiral superfield on S^3 [43].

5.2 $\mathcal{N} = (2, 2)$ SYM partition function

Superfield computation of the partition function of $\mathcal{N} = 2$ SYM on S^3 was carried out in [18]. Here we repeat basic steps of this procedure for the case of $\mathcal{N} = (2, 2)$ SYM on S^2 .

At one-loop order the partition function Z is related to the effective action Γ as follows

$$Z_{\text{SYM}}^{\mathcal{N}=(2,2)} = e^\Gamma. \quad (5.34)$$

To find the effective action we perform the standard background-quantum splitting of the gauge superfield V [44]

$$e^V = e^{\Omega^\dagger} e^{g v} e^\Omega, \quad (5.35)$$

where v is the so-called quantum gauge superfield while Ω is a complex unconstrained prepotential which defines the background gauge superfield V_0 as

$$e^{V_0} = e^{\Omega^\dagger} e^\Omega. \quad (5.36)$$

Upon this splitting the gauge symmetry is realized in two different ways:

$$(i) \quad e^\Omega \rightarrow e^{i\tau} e^\Omega, \quad e^{gv} \rightarrow e^{i\tau} e^{gv} e^{-i\tau}, \quad (5.37)$$

$$(ii) \quad e^\Omega \rightarrow e^{i\lambda} e^\Omega e^{-i\lambda}, \quad e^{gv} \rightarrow e^{i\bar{\lambda}} e^{gv} e^{-i\lambda}. \quad (5.38)$$

Here τ and λ are real and chiral superfield parameters, respectively. The basic idea of the background field method is to fix the gauge symmetry (5.38) keeping the invariance of the effective action under (5.37).

We will compute the one-loop effective action $\Gamma[V_0]$ for the background gauge superfield V_0 taking values in the Cartan subalgebra of the $u(N)$ gauge algebra

$$V_0 = \text{diag}(V_1, V_2, \dots, V_N). \quad (5.39)$$

Moreover, we assume that each of the superfields V_I has constant superfield strengths as in (5.28). In components, such a background is given by (5.12) for every V_I .

The one-loop effective action is defined by the action for quadratic fluctuations around the chosen background. For arbitrary background this action has a conventional form which is similar to the $\mathcal{N} = 1$ $d = 4$ [44] and $\mathcal{N} = 2$ $d = 3$ [45] SYM models

$$S_2 = -\frac{1}{2} \text{tr} \int d^6 z E v (\nabla^\alpha \bar{\nabla}^2 \nabla_\alpha - 4i W^\alpha \nabla_\alpha) v. \quad (5.40)$$

Here the superfield strength W_α and the gauge-covariant derivatives ∇_α and $\bar{\nabla}_\alpha$ are constructed with the use of the background gauge superfield V_0 according to the rules (3.9) and (3.10). Recall that the background superfield V_0 corresponds to the constant scalar superfield strengths G and H while the spinor superfield strengths vanish, $W_\alpha = \bar{W}_\alpha = 0$, see eq. (5.28). For such a background the action (5.40) simplifies to

$$S_2 = -\frac{1}{2} \text{tr} \int d^6 z E v \nabla^\alpha \bar{\nabla}^2 \nabla_\alpha v. \quad (5.41)$$

The operator $\nabla^\alpha \bar{\nabla}^2 \nabla_\alpha$ in (5.41) is degenerate and requires gauge fixing. The gauge symmetry under the λ -transformations (5.38) is fixed by imposing the standard conditions

$$i\bar{\nabla}^2 v = f, \quad i\nabla^2 v = \bar{f}, \quad (5.42)$$

where f is a fixed covariantly chiral superfield, $\bar{\nabla}_\alpha f = 0$.

Following the standard procedure used for quantizing (superfield) gauge theories [44], one should introduce covariantly chiral ghost superfields b and c , $\bar{\nabla}_\alpha b = \bar{\nabla}_\alpha c = 0$. The quadratic part of the ghost superfield action is

$$S_{\text{FP}} = \text{tr} \int d^6 z E (\bar{b}c - b\bar{c}). \quad (5.43)$$

Thus, the one-loop partition function gets the following functional integral representation

$$Z_{\text{SYM}}^{\mathcal{N}=(2,2)} = \int \mathcal{D}v \mathcal{D}b \mathcal{D}c \mathcal{D}\varphi \delta(f - i\bar{\nabla}^2 v) \delta(\bar{f} - i\nabla^2 v) e^{-S_2 - S_{\text{FP}}}. \quad (5.44)$$

Upon averaging this expression with the weight

$$1 = \int \mathcal{D}f \mathcal{D}\varphi e^{\frac{1}{2} \text{tr} \int d^6 z E (\bar{f}f + \bar{\varphi}\varphi)}, \quad (5.45)$$

where f and φ are Grassmann-even and Grassmann-odd chiral superfields, respectively, we end up with the gauge-fixing and the Nielsen-Kallosh ghost superfield actions

$$S_{\text{gf}} = -\frac{1}{4} \text{tr} \int d^6 z E v \{ \nabla^2, \bar{\nabla}^2 \} v, \quad S_{\varphi} = \frac{1}{2} \text{tr} \int d^6 z E \bar{\varphi} \varphi. \quad (5.46)$$

The sum of the actions S_{gf} and S_2 can be recast as follows

$$S_2 + S_{\text{gf}} = -\text{tr} \int d^6 z E v \square_v v, \quad (5.47)$$

where \square_v is the gauge-covariant Laplacian operator acting in the space of general real superfields

$$\square_v = \frac{1}{4} \{ \nabla^2, \bar{\nabla}^2 \} - \frac{1}{2} \nabla^\alpha \bar{\nabla}^2 \nabla_\alpha + 2iW^\alpha \nabla_\alpha. \quad (5.48)$$

Using the algebra of the covariant derivatives (3.4), for an arbitrary gauge superfield background this operator can be written in the form

$$\begin{aligned} \square_v = & -\nabla^a \nabla_a + \left(H + \frac{1}{r} M \right)^2 + \left(G - \frac{i}{2r} R \right)^2 + \frac{1}{2r} [\nabla^\alpha, \bar{\nabla}_\alpha] \\ & + 2iW^\alpha \nabla_\alpha - 2i\bar{W}^\alpha \bar{\nabla}_\alpha - i(\nabla^\alpha W_\alpha). \end{aligned} \quad (5.49)$$

In comparison with the three-dimensional case [18], this operator has additional term with the superfield strength H .

Recall that we consider the gauge superfield background constrained by (5.28). For such a background the form of the operator (5.49) acting in the space of chargeless scalar superfields simplifies to

$$\square_v = -\nabla^a \nabla_a + H^2 + G^2 + \frac{1}{2r} [\nabla^\alpha, \bar{\nabla}_\alpha]. \quad (5.50)$$

After averaging (5.44) with the weight (5.45) the integrals over all superfields become Gaussian and we get the following form of the one-loop partition function of the $\mathcal{N} = (2, 2)$ SYM model

$$Z_{\text{SYM}}^{\mathcal{N}=(2,2)} = \text{Det}^{-1/2} \square_v \cdot Z_\varphi \cdot Z_b \cdot Z_c, \quad (5.51)$$

where Z_φ , Z_b and Z_c are the one-loop partition functions of the chiral ghost superfields.

Let us discuss the contribution to (5.51) of the operator \square_v . In general, as a consequence of the gauge invariance of the effective action, the trace of the logarithm of this operator is given by a functional of superfield strengths G and H

$$-\frac{1}{2} \text{Tr} \ln \square_v = \int d^6 z E \mathcal{L}(G_I, H_I). \quad (5.52)$$

As pointed out in (5.28), we consider the constant superfield strengths G_I and H_I . Hence, the effective Lagrangian $\mathcal{L}(G_I, H_I)$ is also a constant. Therefore the expression (5.52) is

proportional to the volume of the supercoset $\frac{\text{SU}(2|1)}{\text{U}(1) \times \text{U}(1)}$ which vanishes according to (2.15). Thus, the contribution from \square_v to (5.51) is trivial,

$$\text{Det}^{-1/2} \square_v = 1. \quad (5.53)$$

Note that this conclusion is completely analogous to the one for the $\mathcal{N} = 2$ SYM model on S^3 [18].

The equation (5.53) shows that the partition function in the $\mathcal{N} = (2, 2)$ SYM model receives contributions from the ghost superfields only. These are Grassmann-odd chiral superfields in the adjoint representation of the gauge group. It is important to note that the R-charges of these superfields are

$$q(b) = q(c) = 0, \quad q(\varphi) = 2. \quad (5.54)$$

Taking into account these values of the R-charges we apply the formula (5.32) to find the partition functions of the ghost superfields

$$Z_\varphi^{-1} = Z_b = Z_c = \prod_{I < J}^N \left(\frac{(n_I - n_J)^2}{4} + r^2 (\sigma_I - \sigma_J)^2 \right). \quad (5.55)$$

Substituting these partition functions into (5.51) and taking into account (5.53) we find

$$Z_{\text{SYM}}^{\mathcal{N}=(2,2)}(\sigma_I, n_I) = \prod_{I < J}^N \left(\frac{(n_I - n_J)^2}{4} + r^2 (\sigma_I - \sigma_J)^2 \right). \quad (5.56)$$

The one-loop partition function of the $\mathcal{N} = (2, 2)$ SYM model in this form was obtained in [4, 5] using component field computations. Here we re-derived the same result using the superfield method.

An interesting feature of the superfield approach for computing the partition function in the $\mathcal{N} = (2, 2)$ SYM theory is that the result (5.56) appears solely due to the ghost superfields (5.55) while the gauge superfield itself does not contribute (5.53). At first sight this might seem strange since in the ordinary component field computations [4, 5] there are non-trivial contributions both from the ghosts and the fields from the $\mathcal{N} = (2, 2)$ gauge supermultiplet. We stress that there is no contradiction between the component field approach and the superfield method since they give the same result. In fact, this is not surprising because the details of computations depend essentially on the gauge fixing condition. We use the manifestly supersymmetric gauge fixing condition (5.42) while the authors of [4, 5] employed a non-supersymmetric gauge.

5.3 $\mathcal{N} = (4, 4)$ and $\mathcal{N} = (8, 8)$ SYM one-loop partition functions

We will now compute the partition functions of the $\mathcal{N} = (4, 4)$ and $\mathcal{N} = (8, 8)$ SYM models on S^2 described by the actions (4.19) and (4.42), respectively.

The action (4.19) depends on the parameter q which is associated with the R-charge of the chiral superfield Φ that is part of the $\mathcal{N} = (4, 4)$ gauge supermultiplet. So, the partition

function of this model depends not only on the parameters of the Coulomb branch, but also on q

$$Z_{\text{SYM}}^{\mathcal{N}=(4,4)} = Z_{\text{SYM}}^{\mathcal{N}=(4,4)}(n_I, \sigma_I; q). \quad (5.57)$$

Here n_I and σ_I are the parameters which are related to the vacuum values of the scalar fields of the vector multiplet V as in (5.12) and (5.28). Note that in the $\mathcal{N} = (4, 4)$ SYM model we can give vacuum values also to the scalar fields in the chiral multiplet, $\phi_0 = \langle \Phi \rangle$, $\bar{\phi}_0 = \langle \bar{\Phi} \rangle$. However, we simplify the problem by considering vanishing values of these scalars, $\phi_0 = \bar{\phi}_0 = 0$, keeping in mind that the dependence of the partition function on ϕ_0 and $\bar{\phi}_0$ can be easily restored by employing the $\text{SU}(2) \sim \text{SO}(3)$ R-symmetry which rotates $\text{Re}\phi$, $\text{Im}\phi$ and σ .

In comparison with the $\mathcal{N} = (2, 2)$ SYM theory, the partition function of the $\mathcal{N} = (4, 4)$ SYM receives also a contribution from the chiral superfield Φ ,

$$Z_{\text{SYM}}^{\mathcal{N}=(4,4)} = \text{Det}^{-1/2} \left(\square_v - \frac{q}{4r} [\nabla^\alpha, \bar{\nabla}_\alpha] \right) \cdot Z_\varphi \cdot Z_b \cdot Z_c \cdot Z_\Phi, \quad (5.58)$$

where Z_Φ is given in (5.32). Note that in (5.58) the operator \square_v gets shifted by the term $-\frac{q}{4r}[\nabla^\alpha, \bar{\nabla}_\alpha]$ which originates from the second variational derivative of the CS-term in (4.19). Applying the same arguments as in (5.52) and (5.53) to the operator $\square_v - \frac{q}{4r}[\nabla^\alpha, \bar{\nabla}_\alpha]$ one can easily argue that

$$\text{Det}^{-1/2} \left(\square_v - \frac{q}{4r} [\nabla^\alpha, \bar{\nabla}_\alpha] \right) = 1. \quad (5.59)$$

Next, according to (5.55), Z_φ and Z_b cancel against each other, and we end up with the following expression for the $\mathcal{N} = (4, 4)$ SYM partition function

$$Z_{\text{SYM}}^{\mathcal{N}=(4,4)} = Z_c \cdot Z_\Phi, \quad (5.60)$$

where the explicit expressions for Z_c and Z_Φ are given in (5.55) and (5.32), respectively.

In the end of section 4.1.1 we pointed out that the value $q = 0$ in the $\mathcal{N} = (4, 4)$ SYM model is singled out by the requirement that the scalar fields should have non-negative masses. For the vanishing R-charge the factors Z_c and Z_Φ in (5.60) exactly cancel each other and the partition function trivializes

$$Z_{\text{SYM}}^{\mathcal{N}=(4,4)}|_{q=0} = 1. \quad (5.61)$$

A similar trivialization of the one-loop partition function was also observed in [18] for $\mathcal{N} = 4$ SYM on S^3 .

The classical action of the $\mathcal{N} = (8, 8)$ SYM theory (4.42) contains the three chiral superfields Φ_i each of which has the fixed R-charge $q = \frac{2}{3}$. Hence, the expression (5.60) easily generalizes to the case of $\mathcal{N} = (8, 8)$ SYM one-loop partition function

$$Z_{\text{SYM}}^{\mathcal{N}=(8,8)} = Z_c \cdot (Z_\Phi)^3|_{q=\frac{2}{3}}, \quad (5.62)$$

where the expression for Z_c and Z_Φ are given in (5.55) and (5.32), respectively.

6 Localization

6.1 $\mathcal{N} = (2, 2)$ SYM partition function

A representation for the partition function in a general $\mathcal{N} = (2, 2)$ gauge theory which involves the gauge and chiral multiplets was obtained in [4, 5] using the localization method for supersymmetric gauge theories. In this section we discuss how the same representation can be obtained using the superfield form of the $\mathcal{N} = (2, 2)$ SYM action.

Consider the $\mathcal{N} = (2, 2)$ SYM model (3.14) extended with the FI-term (3.21),

$$S = S_{\text{SYM}} + S_{\text{FI}} = -4\text{tr} \int d^6z E \left(\frac{1}{g^2} G^2 - \xi V \right). \quad (6.1)$$

In general [4, 5], one can also extend this action with the topological term

$$S_{\text{top}} = -i \frac{\vartheta}{2\pi} \int \text{tr} F, \quad (6.2)$$

where F is a two-form field strength of the purely bosonic gauge field, $\text{tr} F = \text{tr} dA$, such that ϑ and the Fayet-Iliopoulos coupling constant ξ form a single complex coupling $\tau = \frac{\vartheta}{2\pi} + i\xi$. However, we do not include this term in our consideration since superspace formulation of the action (6.2) is not known.

Before gauge fixing, the partition function in the model (6.1) is given by the functional integral

$$Z = \int \mathcal{D}V e^{-S_{\text{SYM}} - S_{\text{FI}}}. \quad (6.3)$$

In principle, Z can depend on the both couplings $Z = Z(g^2, \xi)$. However, standard localization arguments [3] can be used to demonstrate that Z is independent of g^2 . Indeed, the $\mathcal{N} = (2, 2)$ SYM action is known to be Q -exact with respect to a supersymmetry generator Q on S^2 [4, 5]. Hence, one can harmlessly deform the functional integral (6.3) by introducing an arbitrary real parameter t ,

$$Z(t) = \int \mathcal{D}V e^{-t S_{\text{SYM}} - S_{\text{FI}}}, \quad (6.4)$$

such that Z is in fact independent of t , $\frac{d}{dt} Z(t) = 0$, and, hence, is independent of g^2 as well. Owing to this property, we can compute the functional integral (6.4) in the limit $t \rightarrow \infty$ where some simplifications are expected. Indeed, at large t the functional integral localizes on the critical points V_0 , i.e., on those field configurations which are invariant under the supersymmetry and for which the SYM action vanishes, $S_{\text{SYM}}[V_0] = 0$. In superspace it is easy to find the general solution of the latter equation

$$S_{\text{SYM}} = 0 \quad \Rightarrow \quad W_\alpha = 0, \quad G = G_0 = \text{const}, \quad H = H_0 = \text{const}. \quad (6.5)$$

Indeed, the classical SYM action (3.14) is given by the superfield Lagrangian proportional to G^2 or H^2 integrated over the full superspace. However, according to (2.15), such integrals vanish for constant superfield strengths. Moreover, one can easily see that the superfield background (6.5) is invariant under supersymmetry variations on S^2 which have the general

form (2.32) since the superfields G and H are neutral under the action of the generators R and M . Therefore, in the superfield description, the set of critical points is described by the constant scalar superfield strengths.

For the gauge group $U(N)$, the constants G_0 and H_0 are matrices in the Lie algebra $u(N)$. The standard arguments of residual gauge invariance allow one to reduce the set of these critical points to the Cartan subalgebra of the gauge algebra thus leading to the appearance of the contribution of the Vandermonde determinant into the path integral measure (see, e.g., [46] for a review). However, here we will achieve the same result in a different way. We will show, a posteriori, that the correct expression can be obtained by fixing the background gauge superfield V_0 to belong to the Cartan subalgebra, by imposing this constraint on V_0 “by hand”. In this case the Vandermonde determinant contribution will appear automatically as a part of the one-loop partition function of the $\mathcal{N} = (2, 2)$ SYM theory. This procedure is, in fact, completely analogous to the one given in [18] for the superfield gauge theory on S^3 , but we repeat its basic steps here for completeness.

Let us start by considering the gauge superfield background (6.5) without additional restrictions. In the path integral (6.4) we perform the background-quantum splitting $V \rightarrow \left(V_0, \frac{1}{\sqrt{t}}v'\right)$ similar to (5.35), but using the parameter t instead of the gauge coupling constant,

$$e^V = e^{\Omega^\dagger} e^{\frac{1}{\sqrt{t}}v'} e^\Omega, \quad e^{V_0} = e^{\Omega^\dagger} e^\Omega. \quad (6.6)$$

Upon this splitting we assume that the space of all the fields $\{V\}$ is a direct sum of the spaces of the fields $\{V_0\}$ and $\{v'\}$. Then, the integration measure of the functional integral factorizes

$$\mathcal{D}V = \mathcal{D}V_0 \mathcal{D}v'. \quad (6.7)$$

This means that the modes which are taken into account by $\mathcal{D}V_0$ should be absent in the measure $\mathcal{D}v'$. Recall that the value of the gauge superfield V_0 is related to the constant scalar gauge superfield strengths (6.5). Hence, in the measure $\mathcal{D}v'$ the integration goes over such superfields which have *non-constant* superfield strengths. We denote the space of these superfields by $\{v'\}$ to distinguish them from the unconstrained superfields $\{v\}$.

Following the same steps as in section 5.2, upon the background-quantum splitting (6.6) and fixing the gauge freedom for the superfield v' we get the following representation for the path integral (6.4)

$$Z(t) = \int \mathcal{D}V_0 \mathcal{D}v' \mathcal{D}b \mathcal{D}c \delta(f - i\bar{\nabla}^2 v') \delta(\bar{f} - i\nabla^2 v') e^{-tS_{\text{SYM}}[V_0, \frac{1}{\sqrt{t}}v'] - S_{\text{FI}}[V_0, \frac{1}{\sqrt{t}}v'] - S_{\text{FP}}}. \quad (6.8)$$

The part of the Faddeev-Popov action S_{FP} which is quadratic in superfields has the form (5.43).

The basic idea of the localization method is to compute the functional integral (6.8) in the limit $t \rightarrow \infty$ in which only quadratic fluctuations of the superfields around the

background V_0 survive,

$$\begin{aligned} -tS_{\text{SYM}} \left[V_0, \frac{1}{\sqrt{t}}v' \right] &= -S_2 [V_0, v'] + O(1/\sqrt{t}), \\ -S_{\text{FI}} \left[V_0, \frac{1}{\sqrt{t}}v' \right] &= -S_{\text{FI}}[V_0] + O(1/\sqrt{t}), \end{aligned} \quad (6.9)$$

where the action $S_2[V_0, v']$ is given by (5.41). Thus, sending t to infinity, we get the following representation for the partition function (6.8)

$$Z = \lim_{t \rightarrow \infty} Z(t) = \int \mathcal{D}V_0 e^{-S_{\text{FI}}[V_0]} \cdot Z'_{\text{SYM}}, \quad (6.10)$$

where

$$Z'_{\text{SYM}} = \int \mathcal{D}v' \mathcal{D}b \mathcal{D}c \mathcal{D}\varphi \delta(f - i\bar{\nabla}^2 v') \delta(\bar{f} - i\nabla^2 v') e^{-S_2[V_0, v'] - S_{\text{FP}}} \quad (6.11)$$

is the one-loop $\mathcal{N} = (2, 2)$ SYM partition function which is very similar to (5.44), but with the restriction on the superfields v' such that they do not include the zero modes corresponding to the constant scalar superfield strengths. Recall that these modes are taken into account by the measure $\mathcal{D}V_0$ according to (6.7).

With the use of the superfield methods it is difficult to compute the functional integral (6.11) because of the constraint on the integration values of the superfield v' . However, one can rearrange the measure of the functional integral (6.10) in such a way that the integration over v' becomes unconstrained. Recall that the background superfield V_0 is the Lie-algebra-valued matrix corresponding to the constant superfield strengths (6.5). This matrix can be naturally decomposed as

$$V_0 = V_0^{\mathfrak{h}} + V_0^{\mathfrak{r}}, \quad V_0^{\mathfrak{h}} \in \mathfrak{h}, \quad V_0^{\mathfrak{r}} \in \mathfrak{r}, \quad (6.12)$$

where the Lie algebra \mathfrak{g} is given by the direct sum of the Cartan subalgebra \mathfrak{h} and the root space directions \mathfrak{r} , $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$. Thus, the integration measure $\mathcal{D}V_0$ decomposes as

$$\mathcal{D}V_0 = \mathcal{D}V_0^{\mathfrak{h}} \mathcal{D}V_0^{\mathfrak{r}}. \quad (6.13)$$

Now, we combine the measures $\mathcal{D}V_0^{\mathfrak{r}}$ and $\mathcal{D}v'$ together

$$\mathcal{D}v = \mathcal{D}V_0^{\mathfrak{r}} \mathcal{D}v' \quad (6.14)$$

such that the new measure $\mathcal{D}v$ includes the missing zero modes of fields v' and the superfield v becomes unconstrained.² With this rearrangement of the integration measure in (6.11) we end up with the following expression for the partition function

$$Z = \int \mathcal{D}V_0^{\mathfrak{h}} e^{-S_{\text{FI}}[V_0^{\mathfrak{h}}]} \cdot Z_{\text{SYM}}[V_0^{\mathfrak{h}}]. \quad (6.15)$$

²Note that without loss of generality the superfields v and v' can be considered to belong to the space \mathfrak{r} orthogonal to the Cartan subalgebra of the gauge algebra since the corresponding Cartan components of these fields do not interact with $V_0^{\mathfrak{h}}$.

In this expression the functional integration is performed over the background superfield V_0^h taking values in the Cartan subalgebra of the gauge algebra and $Z_{\text{SYM}}[V_0^h]$ is precisely the $\mathcal{N} = (2, 2)$ SYM partition function (5.56).

Note that we could arrive at the representation for the partition function (6.15) by imposing the constraint on V_0 to belong to the Cartan subalgebra from the very beginning. In this case we do not need to care about the Vandermonde determinant contribution to the functional integral because it is automatically taken into account in $Z_{\text{SYM}}[V_0^h]$.

For the gauge superfield background (5.27), (5.28) each of the superfields V_I is given in components by (5.12). Every V_I in components has just two degrees of freedom given by the real variable σ_I corresponding to the vacuum expectation value of the scalar σ and by an integer n_I which is related to the vacuum expectation value of another scalar η . Thus, the integration measure of the functional integral (6.15) can be rewritten as

$$\int \mathcal{D}V_0^h \rightarrow \int \prod_{I=1}^N d\sigma_I \sum_{\text{all } n_I} . \quad (6.16)$$

In other words, one should integrate over all the continuous parameters σ_I and sum over all the integers n_I .

Using (3.33), one can bring the FI-term in the functional integral (6.15) to the following form

$$S_{\text{FI}}[V_0] = \xi \int d^2x \sqrt{h} \text{tr} D = \frac{i\xi}{r} \sum_{I=1}^N \sigma_I \text{Vol}(S^2) = 4\pi i r \xi \sum_{I=1}^N \sigma_I . \quad (6.17)$$

Finally, substituting (6.17) and (5.56) into (6.15) we end up with

$$Z = \int \prod_{I=1}^N d\sigma_I \sum_{\text{all } n_I} e^{-4\pi i r \xi \sum_{I=1}^N \sigma_I} \prod_{I < J}^N \left(\frac{(n_I - n_J)^2}{4} + r^2 (\sigma_I - \sigma_J)^2 \right) . \quad (6.18)$$

In this form the partition function in the $\mathcal{N} = (2, 2)$ gauge theories was obtained in [4, 5] using component field methods for computing one-loop determinants. Here we re-derived the same result starting with a superfield formulation of this model.

Note also that, in general, the exponential of the topological term (6.2) can be inserted into the integral in (6.18), and also the contributions of chiral matter multiplets can be taken into account. All these cases were studied in [4, 5].

6.2 Gaiotto-Witten and ABJ(M) models reduced to S^2

The classical actions of Gaiotto-Witten (4.51) and ABJM models (4.65) are very similar, so their partition functions can be constructed using the same procedure which mimics the one for the corresponding three-dimensional theories [43, 47]. The essential difference among these models is that the ABJM model has twice as many chiral superfields that give extra contributions. Recall that we denote the chiral superfield as X_{\pm} while the gauge superfields are V and \tilde{V} . We consider the gauge group $U(M) \times U(N)$.

Before gauge fixing, the partition function in the Gaiotto-Witten or ABJM models is represented by the functional integral

$$Z = \int \mathcal{D}X_{\pm} \mathcal{D}V \mathcal{D}\tilde{V} e^{-S[X,V,\tilde{V}]}, \quad (6.19)$$

where $S[X, V, \tilde{V}]$ is either S_{GW} or S_{ABJM} . We deform this partition function by inserting the Q -exact $\mathcal{N} = (2, 2)$ SYM action (3.14) for the both gauge superfields multiplied by a real parameter t

$$Z(t) = \int \mathcal{D}X_{\pm} \mathcal{D}V \mathcal{D}\tilde{V} e^{-S[X,V,\tilde{V}] - tS_{\text{SYM}}[V] - tS_{\text{SYM}}[\tilde{V}]}. \quad (6.20)$$

For large t the functional integral over the gauge superfields localizes on the critical points V_0 and \tilde{V}_0 which are described by the superfield equations (6.5) for each of the gauge superfields. As has been explained in the previous subsection, we can further restrict these superfields to belong to the Cartan subalgebra

$$V_0 = \text{diag}(V_1, V_2, \dots, V_M), \quad \tilde{V}_0 = \text{diag}(\tilde{V}_1, \tilde{V}_2, \dots, \tilde{V}_N). \quad (6.21)$$

Each of V_I and \tilde{V}_J contains component fields with values as in eq. (5.12), i.e., the background is described by the pairs (n_I, σ_I) and $(\tilde{n}_J, \tilde{\sigma}_J)$ corresponding to vevs of the scalars in the vector multiplets.

Similar to (6.6), we perform the background-quantum splitting for V and \tilde{V} in (6.20)

$$V \rightarrow \left(V_0, \frac{1}{\sqrt{t}}v\right), \quad \tilde{V} \rightarrow \left(\tilde{V}_0, \frac{1}{\sqrt{t}}\tilde{v}\right). \quad (6.22)$$

For large t it is sufficient to consider only quadratic fluctuations in the SYM actions while the action $S[X, V, \tilde{V}]$ should be considered for purely background gauge superfields,

$$Z = \lim_{t \rightarrow \infty} Z(t) = \int \mathcal{D}X_{\pm} \mathcal{D}V_0 \mathcal{D}v \mathcal{D}\tilde{V}_0 \mathcal{D}\tilde{v} e^{-S[X, V_0, \tilde{V}_0] - S_2[V_0, v] - S_2[\tilde{V}_0, \tilde{v}]}, \quad (6.23)$$

where the action S_2 is given by (5.41). Upon gauge fixing the transformations of superfields v and \tilde{v} in the standard way (5.42), we get the following representation for the partition function

$$Z = \int \mathcal{D}V_0 \mathcal{D}\tilde{V}_0 e^{-S_{\text{CS}}[V_0] + S_{\text{CS}}[\tilde{V}_0] - S_{\text{FI}}[V_0 + \tilde{V}_0]} \cdot Z_X \cdot Z_{\text{SYM}}[V_0] \cdot Z_{\text{SYM}}[\tilde{V}_0], \quad (6.24)$$

where $Z_{\text{SYM}}[V_0]$ and $Z_{\text{SYM}}[\tilde{V}_0]$ have the form (5.56) while Z_X is the one-loop partition function for the (anti)chiral superfields X_{\pm} . The term $S_{\text{FI}}[V_0 + \tilde{V}_0]$ is given by (4.53).

Recall that in the Gaiotto-Witten model we have one chiral superfield X_+ in the bi-fundamental representation and another chiral scalar X_- in the anti-bi-fundamental representation while in the ABJM model the number of these superfields is doubled. Hence, the one-loop partition function Z_X is a simple generalization of (5.32):

$$Z_X = \prod_{I,J} \left[\frac{\Gamma\left(\frac{q_+}{2} + \frac{|n_I - \tilde{n}_J|}{2} + ir(\sigma_I - \tilde{\sigma}_J)\right) \Gamma\left(\frac{q_-}{2} + \frac{|n_I - \tilde{n}_J|}{2} - ir(\sigma_I - \tilde{\sigma}_J)\right)}{\Gamma\left(1 - \frac{q_+}{2} + \frac{|n_I - \tilde{n}_J|}{2} - ir(\sigma_I - \tilde{\sigma}_J)\right) \Gamma\left(1 - \frac{q_-}{2} + \frac{|n_I - \tilde{n}_J|}{2} + ir(\sigma_I - \tilde{\sigma}_J)\right)} \right]^p, \quad (6.25)$$

where $p = 1$ for the Gaiotto-Witten model and $p = 2$ for ABJM.

It is easy to find the values of the CS- and FI-terms in the Gaiotto-Witten and ABJM actions in a form similar to eq. (6.17)

$$S_{\text{CS}}[V_0] - S_{\text{CS}}[\tilde{V}_0] = i\pi\kappa \sum_{I=1}^M \left(\frac{n_I^2}{4} - \sigma_I^2 r^2 \right) - i\pi\kappa \sum_{J=1}^N \left(\frac{\tilde{n}_J^2}{4} - \tilde{\sigma}_J^2 r^2 \right), \quad (6.26)$$

$$S_{\text{FI}}[V_0 + \tilde{V}_0] = -\frac{\pi}{4}\kappa r(q_+ - q_-) \left(\sum_{I=1}^M \sigma_I + \sum_{J=1}^N \tilde{\sigma}_J \right). \quad (6.27)$$

We substitute these expressions into (6.24) and take into account that the functional measure reduces to conventional integrations over all σ_I and $\tilde{\sigma}_J$ and sums over all n_I and \tilde{n}_J according to (6.16)

$$\begin{aligned} Z &= \int \prod_{I=1}^M d\sigma_I \sum_{\text{all } n_I} \int \prod_{J=1}^N d\tilde{\sigma}_J \sum_{\text{all } \tilde{n}_J} Z_X \cdot Z_{\text{SYM}}^{\mathcal{N}=(2,2)}(\sigma_I, n_I) \cdot Z_{\text{SYM}}^{\mathcal{N}=(2,2)}(\tilde{\sigma}_J, \tilde{n}_J) \\ &\times \exp \left\{ -i\pi\kappa \sum_{I=1}^M \left(\frac{n_I^2}{4} - \sigma_I^2 r^2 + \frac{i}{4}(q_+ - q_-)r\sigma_I \right) \right\} \\ &\times \exp \left\{ i\pi\kappa \sum_{J=1}^N \left(\frac{\tilde{n}_J^2}{4} - \tilde{\sigma}_J^2 r^2 - \frac{i}{4}(q_+ - q_-)r\tilde{\sigma}_J \right) \right\}. \end{aligned} \quad (6.28)$$

Here Z_X and $Z_{\text{SYM}}^{\mathcal{N}=(2,2)}(\sigma_I, n_I)$ are given by (6.25) and (5.56), respectively.

By shifting the integration variables

$$\sigma_I \rightarrow \sigma_I - \frac{i}{8r}(q_+ - q_-), \quad \tilde{\sigma}_J \rightarrow \tilde{\sigma}_J + \frac{i}{8r}(q_+ - q_-) \quad (6.29)$$

the expression (6.28) can be slightly simplified

$$\begin{aligned} Z &= \int \prod_{I=1}^M d\sigma_I \sum_{\text{all } n_I} \int \prod_{J=1}^N d\tilde{\sigma}_J \sum_{\text{all } \tilde{n}_J} Z_{\text{SYM}}^{\mathcal{N}=(2,2)}(\sigma_I, n_I) \cdot Z_{\text{SYM}}^{\mathcal{N}=(2,2)}(\tilde{\sigma}_J, \tilde{n}_J) \\ &\times \exp \left\{ -i\pi\kappa \sum_{I=1}^M \left(\frac{n_I^2}{4} - \sigma_I^2 r^2 \right) + i\pi\kappa \sum_{J=1}^N \left(\frac{\tilde{n}_J^2}{4} - \tilde{\sigma}_J^2 r^2 \right) + \frac{i\pi\kappa}{16}(N - M)(q_+ - q_-)^2 \right\} \\ &\times \prod_{I,J} \left[\frac{\Gamma \left(\frac{q_+ + q_-}{4} + \frac{|n_I - \tilde{n}_J|}{2} + ir(\sigma_I - \tilde{\sigma}_J) \right) \Gamma \left(\frac{q_+ + q_-}{4} + \frac{|n_I - \tilde{n}_J|}{2} - ir(\sigma_I - \tilde{\sigma}_J) \right)}{\Gamma \left(1 - \frac{q_+ + q_-}{4} + \frac{|n_I - \tilde{n}_J|}{2} - ir(\sigma_I - \tilde{\sigma}_J) \right) \Gamma \left(1 - \frac{q_+ + q_-}{4} + \frac{|n_I - \tilde{n}_J|}{2} + ir(\sigma_I - \tilde{\sigma}_J) \right)} \right]^p. \end{aligned} \quad (6.30)$$

One can see that when the ranks of the gauge groups are equal, $M = N$, the charges of the chiral superfields q_+ and q_- enter the partition function in the combination $q_+ + q_-$. Therefore, for $M = N$ without loss of generality we can assume that

$$q := q_+ = q_-. \quad (6.31)$$

Another important observation is that upon re-scaling the integration variables $\sigma_I \rightarrow \frac{1}{r}\sigma_I$ and $\tilde{\sigma}_J \rightarrow \frac{1}{r}\tilde{\sigma}_J$ the partition function becomes independent of the radius of the sphere. So,

putting for simplicity $r=1$ and $M=N$, the partition functions of the Gaiotto-Witten ($p=1$) and ABJM ($p=2$) models which follow from eq. (6.30) get the following explicit form

$$Z_{\text{GW}} = \int \left(\prod d\sigma_I \right) \sum_{\text{all } n_I} \int \left(\prod d\tilde{\sigma}_J \right) \sum_{\text{all } \tilde{n}_J} \quad (6.32)$$

$$\begin{aligned} & \times \exp \left\{ -i\pi\kappa \sum \left(\frac{n_I^2}{4} - \frac{\tilde{n}_I^2}{4} - \sigma_I^2 + \tilde{\sigma}_I^2 \right) \right\} \\ & \times \prod_{I<J} \left(\frac{(n_I - n_J)^2}{4} + (\sigma_I - \sigma_J)^2 \right) \prod_{I<J} \left(\frac{(\tilde{n}_I - \tilde{n}_J)^2}{4} + (\tilde{\sigma}_I - \tilde{\sigma}_J)^2 \right) \\ & \times \prod_{I,J} \frac{\Gamma \left(\frac{q}{2} + \frac{|n_I - \tilde{n}_J|}{2} + i(\sigma_I - \tilde{\sigma}_J) \right) \Gamma \left(\frac{q}{2} + \frac{|n_I - \tilde{n}_J|}{2} - i(\sigma_I - \tilde{\sigma}_J) \right)}{\Gamma \left(1 - \frac{q}{2} + \frac{|n_I - \tilde{n}_J|}{2} - i(\sigma_I - \tilde{\sigma}_J) \right) \Gamma \left(1 - \frac{q}{2} + \frac{|n_I - \tilde{n}_J|}{2} + i(\sigma_I - \tilde{\sigma}_J) \right)}, \end{aligned}$$

$$Z_{\text{ABJM}} = \int \left(\prod d\sigma_I \right) \sum_{\text{all } n_I} \int \left(\prod d\tilde{\sigma}_J \right) \sum_{\text{all } \tilde{n}_J} \quad (6.33)$$

$$\begin{aligned} & \times \exp \left\{ -i\pi\kappa \sum \left(\frac{n_I^2}{4} - \frac{\tilde{n}_I^2}{4} - \sigma_I^2 + \tilde{\sigma}_I^2 \right) \right\} \\ & \times \prod_{I<J} \left(\frac{(n_I - n_J)^2}{4} + (\sigma_I - \sigma_J)^2 \right) \prod_{I<J} \left(\frac{(\tilde{n}_I - \tilde{n}_J)^2}{4} + (\tilde{\sigma}_I - \tilde{\sigma}_J)^2 \right) \\ & \times \prod_{I,J} \left[\frac{\Gamma \left(\frac{1}{4} + \frac{|n_I - \tilde{n}_J|}{2} + i(\sigma_I - \tilde{\sigma}_J) \right) \Gamma \left(\frac{1}{4} + \frac{|n_I - \tilde{n}_J|}{2} - i(\sigma_I - \tilde{\sigma}_J) \right)}{\Gamma \left(\frac{3}{4} + \frac{|n_I - \tilde{n}_J|}{2} - i(\sigma_I - \tilde{\sigma}_J) \right) \Gamma \left(\frac{3}{4} + \frac{|n_I - \tilde{n}_J|}{2} + i(\sigma_I - \tilde{\sigma}_J) \right)} \right]^2. \end{aligned}$$

In (6.33) we have also taken into account that the R-charges of chiral superfields in the ABJM model are fixed to be $q_+ = q_- = \frac{1}{2}$.

In [48] the exact partition function in the ABJM model was used to show that at large N its free energy scales as $N^{3/2}$ which is in agreement with the number of the degrees of freedom of a system of multiple M2 branes. It would be interesting to study a similar problem for the ABJM model reduced to S^2 and to understand which system of branes it describes.

7 Discussion

To summarize, in this paper we have elaborated on a superfield approach based on the supercoset $\frac{\text{SU}(2|1)}{\text{U}(1) \times \text{U}(1)}$ for studying classical and quantum aspects of supersymmetric field theories on S^2 . We constructed the supersymmetric Cartan forms, supercurvature, super-torsion and supercovariant derivatives on this coset and applied them for constructing classical actions for gauge and chiral superfields.

We have also given classical actions for various models with extended supersymmetry on S^2 in terms of the $\mathcal{N} = (2,2)$ superfields. Among them, there are the actions for the $\mathcal{N} = (4,4)$ hypermultiplet, $\mathcal{N} = (4,4)$ and $\mathcal{N} = (8,8)$ SYM models as well as the actions for the Gaiotto-Witten and ABJM theories reduced to S^2 . For all these models we have derived the transformations of hidden supersymmetries realized on the $\mathcal{N} = (2,2)$ superfields. To the best of our knowledge, the classical superfield actions for the models with extended supersymmetry on S^2 have not been considered before.

We have demonstrated that the superfield method facilitates the computation of the partition functions of supersymmetric gauge and matter theories on S^2 and helps finding critical points in the space of fields for the localization technique. In particular, we have re-derived the known expressions for the one-loop partition functions found originally in [4, 5] for the $\mathcal{N} = (2, 2)$ SYM and the chiral superfield models. An advantage of the superfield method is that the cancellations among bosonic and fermionic contributions to the one-loop determinants of the quadratic fluctuations occur automatically. We have also demonstrated how the localization technique applies to the superfield description of the $\mathcal{N} = (2, 2)$ SYM model which was originally considered in [4, 5]. A new result of this paper is the expression for the partition functions of the Gaiotto-Witten and ABJM models reduced to S^2 . For these models the localization formula is very similar to the one for the corresponding models on S^3 [43, 47] and differs from it mainly by the form of the one-loop determinants for the chiral and gauge multiplets. It would be instructive to study the large N behavior of the partition function in the ABJM model reduced to S^2 and compare it with the corresponding S^3 partition function [48]. It would also be of interest to elaborate on peculiarities of the superconformal structure of the S^2 -counterparts of the Gaiotto-Witten and ABJM models in comparison with the S^3 case [18, 49].

It would be very natural to extend our results to the superfield models in higher-dimensional ($d \geq 4$) curved backgrounds.³ However, already in $d = 4$ the minimal supersymmetry on the four-sphere is $\mathcal{N} = 2$, and it is well known that the quantization of $\mathcal{N} = 2$ SYM and hypermultiplets keeping supersymmetry off-shell requires special methods such as the use of harmonic superspace [52–55]. It is very tempting to extend harmonic superspace techniques to the case of superfield models on the sphere or in the AdS space.

Another possible extension of the results of this paper could be the consideration of twisted chiral and vector $\mathcal{N} = (2, 2)$ supermultiplets. As was demonstrated in [8, 9], quantum partition functions of such models compute the exact Kähler potential for Calabi-Yau target space of $\mathcal{N} = (2, 2)$ non-linear sigma-models. In superspace, classical actions for these models were systematically studied in [11]. It would be of interest to develop a superfield approach for computing partition functions of these models. This issue becomes even more intriguing for the two-dimensional models with extended supersymmetry on S^2 . Indeed, as was pointed out in earlier papers [56–59], there are many inequivalent versions of twisted multiplets with $(4, 4)$ supersymmetry in flat superspace. Assuming that these models allow for a superfield description in the curved superspace based on the supercoset $\frac{\text{SU}(2|1)}{\text{U}(1) \times \text{U}(1)}$, it is tempting to understand the difference among these models on the quantum level by comparing their partition functions. These problems require a separate systematic study.

The papers [4, 5] showed that the two-dimensional supersymmetric theories exhibit rich quantum dynamics with many non-trivial dualities. This motivates further study of low-energy dynamics of these models and, in particular, their low-energy effective actions. Note that the low-energy effective actions in three-dimensional gauge and matter theories in the flat $\mathcal{N} = 2$ superspace were derived in [45, 60–64].

³Quantum mechanical (i.e. $d = 1$) models on different cosets of $\text{SU}(2|1)$ were considered in [50, 51].

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